From mean-variance analysis to mental accounting and back: Bridging contributions of Markowitz to portfolio selection*

Alexandre M. Baptista**

The George Washington University

October 21, 2024

*This article has benefited from previous joint work with Gordon J. Alexander and Shu Yan. However, the views in the article are those of the author and do not necessarily represent their concurrence. The article has also benefited from the comments of session participants at the 2024 SEINFORMS Meeting.
**Address: The George Washington University, School of Business, 2201 G Street, NW, Washington, DC 20008. Tel.: 202-994-3309. E-mail: alexbapt@gwu.edu.

From mean-variance analysis to mental accounting and back: Bridging contributions of Markowitz to portfolio selection

Abstract

Modern portfolio theory considers investors whose preferences are often represented by a function defined over the means and variances of portfolio returns and specified with a risk aversion coefficient; see Markowitz (1952). In Das et al. (2010), however, Markowitz and three co-authors consider an investor who has: (i) mental accounts (hereafter, 'accounts') with different investing motives; (ii) preferences within an account that are specified by some maximum threshold probability of the account's return being less than or equal to some threshold return; and (iii) threshold probabilities and threshold returns (hereafter, 'thresholds') that possibly vary across accounts. Like the optimal portfolio in Markowitz's model, optimal portfolios within accounts in Das et al.'s model are on the mean-variance frontier and so is the corresponding aggregate portfolio if short selling is allowed. Bridging Markowitz's contributions, several analytical results are here uncovered. These results encompass: (1) the losses arising from misspecifying a risk aversion coefficient in Markowitz's model; (2) the impact of using thresholds instead of a risk aversion coefficient to specify investor preferences on portfolio selection; and (3) the risk aversion coefficients of investors in Markowitz's model who select the same portfolios as an investor in Das et al.'s model selects within accounts.

Keywords: modern portfolio theory; mean-variance analysis; mental accounting; tail risk; behavioral finance.

JEL Classification: G11

1. Introduction

The origins of modern portfolio theory can be traced to the seminal paper of Markowitz (1952). In his model, investors assess portfolios based on their expected rewards and risks.[1] The expected reward of a portfolio is measured by the *mean* of the portfolio's future return (hereafter, 'mean' or 'expected return'). The risk of a portfolio is measured by the *variance* or standard deviation of the portfolio's future return (hereafter, respectively, 'variance' and 'standard deviation'). Since Markowitz considers investors who prefer the portfolio with the smallest variance among portfolios with the same mean, they select portfolios on the *mean-variance frontier*. A portfolio is on the mean-variance frontier if there is no portfolio with the same mean and a smaller variance. In order to locate an investor's optimal portfolio on this frontier, the investor's preferences are often represented by a function defined over the means and variances of portfolios and specified with a risk aversion coefficient.[2]

In Das et al. (2010), however, Markowitz and three co-authors note that modern portfolio theory does not address the practical fact that an investor might have: (i) various investing motives such as retirement and bequest;[3] (ii) preferences that are difficult to correctly specify with a risk aversion coefficient;[4] and (iii) different goals for different investing motives. Accordingly, their model considers an investor whose wealth is divided among mental accounts (hereafter, 'accounts') with different investing motives. The investor's preferences within a

[1] Markowitz (1959) reviews the model of Markowitz (1952), whereas Markowitz (2000, 2008) collects his work. For insightful perspectives on his contributions, see Rubinstein (2002), Goetzmann (2023), and the 2024 Special Issue of *The Journal of Portfolio Management* dedicated to him.

[2] While Markowitz is often viewed as the father of modern portfolio theory, Markowitz (1999, p. 5) notes that "*Roy (1952) can claim an equal share of this honor.*" The investor in Roy's model selects the portfolio that maximizes the ratio of (a) excess expected return over some disastrous level of return to (b) standard deviation. For a review of this model, see Elton et al. (2014, pp. 226–229).

[3] In practice, money management firms list many investing motives (e.g., college education, a house, a wedding, and a vacation); see <investor.vanguard.com/investor-resources-education/investing-goals>.

[4] Das et al. note two reasons for investors not being able to correctly specify their risk aversion coefficients. First, since an investor might have different risk aversion coefficients for different investment motives, the investor might not be able to properly weigh such coefficients to find the overall risk aversion coefficient used to identify the portfolio that in aggregate maximizes the investor's overall satisfaction. Second, investors might find that the specification of their levels of risk aversion in units of variance is not intuitive.

given account are specified by a threshold probability and a threshold return (hereafter, 'thresholds').[5] Formally, the optimal portfolio within the account maximizes the account's expected return subject to the constraint that the probability of the account's return being less than or equal to the threshold return does not exceed the threshold probability. Hence, the account's Value-at-Risk (VaR) at the confidence level of one minus the threshold probability cannot exceed minus one multiplied by the threshold return.[6] Recognizing that different accounts have differ investment motives, thresholds possibly vary across accounts.[7]

The models of Markowitz and Das et al. are closely related if asset returns are assumed to have a multivariate Normal distribution and thresholds are such that optimal portfolios within accounts exist in Das et al.'s model. Since the VaR of a portfolio is a linear function of its mean and standard deviation under this distribution, optimal portfolios within accounts are on the mean-variance frontier. Hence, the optimal portfolio within a given account of an investor in Das et al.'s model would be selected by a hypothetical investor in Markowitz's model with some (implied) risk aversion coefficient. Conversely, the optimal portfolio of an investor in the latter model would be selected within an account of a hypothetical investor in the former with some (implied) thresholds. The aggregate portfolio of an investor in Das et al.'s model (the combination of the investor's optimal portfolios within accounts) is also on the mean-variance frontier if short selling is allowed.[8]

[5] In practice, financial advisers (working independently or at money management firms) utilize financial advising programs that use thresholds to reflect the goals of investors; see Statman (2017, pp. 208–217).

[6] Since the practical use of VaR is at the heart of modern risk management (see Hull (2023)) and the theoretical use of VaR as a measure of risk is related to mean-variance analysis under certain conditions (see Baumol (1963) and Alexander and Baptista (2002)), Alexander (2009) contends that Markowitz is also the father of modern risk management.

[7] The model of Das et al. extends the model of Telser (1955) from the single-account case to the multiple-account case. While Das et al. (2011) and Statman (2024) review the former model, Elton et al. (2014, pp. 230–231) review the latter; see also Arzac and Bawa (1977).

[8] The investor in the behavioral portfolio selection model of Shefrin and Statman (2000) has one or more accounts and is possibly risk seeking. In comparison, the investor in Das et al.'s model has two or more accounts and is risk averse. Hence, Das et al.'s model integrates features of behavioral portfolio theory (accounts) and modern portfolio theory (optimal portfolios within accounts are on the mean-variance frontier and so is the aggregate portfolio if short selling allowed). When short selling is disallowed, Das et al. show that the former portfolios are also on the frontier but the latter might lie away from but close to the frontier.

Bridging Markowitz's contributions, results along three dimensions are here uncovered.[9] First, consider the loss in certainty-equivalent return (CER) arising from an investor using an erroneous (instead of the 'true') risk aversion coefficient in seeking to find the optimal portfolio in Markowitz's model. An analytical expression for the loss in CER is derived. Importantly, the loss in CER in the case where the erroneous coefficient is less than the 'true' coefficient by some amount *exceeds* the loss in CER in the case where the erroneous coefficient is greater than the 'true' coefficient by the same amount. Moreover, the loss in CER is unbounded (bounded) from above in the former (latter) case.

Second, consider the impact of using thresholds instead of a risk aversion coefficient to specify investor preferences on portfolio selection. For any positive risk aversion coefficient, the optimal portfolio in Markowitz's model lies on the mean-variance frontier above the global minimum-variance portfolio. In comparison, for any pair of thresholds such that the optimal portfolio within a given account exists in Das et al.'s model, this portfolio lies on the mean-variance frontier at the same point as or above the global minimum-VaR portfolio at the confidence level of one minus the threshold probability, which in turn lies above the global minimum-variance portfolio. While different risk aversion coefficients lead to different optimal portfolios in Markowitz's model, infinitely many pairs of thresholds lead to the same optimal portfolio within an account in Das et al.'s model.

Third, consider the risk aversion coefficients implied by optimal portfolios within accounts (when these portfolios exist). Using an analytical characterization of such coefficients, key observations are made. Assuming thresholds such that the optimal portfolio within a given account exists, the risk aversion coefficient implied by this portfolio: (i) is strictly between

[9] There is a vast literature on extensions of the models of Markowitz and Das et al. (involving consideration of, for example, portfolio delegation, background risk, estimation risk, and equilibrium). For a review of such extensions, see Fabozzi et al. (2010), Alexander et al. (2020), Koumou (2020), and references therein.

zero and a positive value associated with the global minimum-VaR portfolio at the confidence level of one minus the threshold probability; (ii) decreases in the threshold probability; (iii) increases in the threshold return; and (iv) is very sensitive to the thresholds if the coefficient is relatively large but notably less so if the coefficient is relatively small.

The paper proceeds as follows. Section 2 characterizes the composition of the optimal portfolio of an investor in Markowitz's model and the loss in CER arising from the misspecification of the investor's risk aversion coefficient. Section 3 characterizes the existence and composition of the optimal portfolios within accounts and the aggregate portfolio of an investor in Das et al.'s model as well as the risk aversion coefficients implied by the former and latter portfolios. Section 4 uses a numerical example to illustrate the theoretical results uncovered in Sections 2 and 3. Section 5 concludes.[10]

2. The model of Markowitz (1952)

This section examines the model of Markowitz (1952).

2.1. Assumptions

Suppose that a risk-free asset is not available for trade.[11] Let $N > 1$ be the number of risky assets that are available for trade. The first two moments of their return distribution are assumed to be finite. Let $\boldsymbol{\mu}$ be the $N \times 1$ vector of their expected returns. Suppose that $rank([\mathbf{1}_N \quad \boldsymbol{\mu}]) = 2$ where $\mathbf{1}_N$ is the $N \times 1$ unit vector.[12] Let $\boldsymbol{\Sigma}$ be the $N \times N$ variance-covariance matrix for asset returns. Suppose that $rank(\boldsymbol{\Sigma}) = N$.[13] Hereafter, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are referred to as *optimization inputs*.

[10] An Online Appendix contains: (i) a table listing the notation used for the models of Markowitz and Das et al.; and (ii) proofs of the theoretical results in the paper.

[11] Tobin (1958) extends the results in Markowitz's model to the case where a risk-free asset is assumed to be available for trade. Sharpe (1964) characterizes expected asset returns in equilibria for economies where investors use Tobin's model for portfolio selection.

[12] The case where all assets have the *same* expected return is thus precluded.

[13] The existence of a combination of risky assets with a risk-free return is thus precluded.

A portfolio is an $N \times 1$ vector $\boldsymbol{w}$ with $\boldsymbol{w}'\mathbf{1}_N = 1$. The nth entry of portfolio $\boldsymbol{w}$ is the weight of asset n in the portfolio. A positive (negative) weight represents a long (short) position. Here, short selling is allowed and asset weights are assumed to be unbounded.[14]

Let $r_{\boldsymbol{w}}$ denote the random return of portfolio $\boldsymbol{w}$. Its mean or expected return is $E[r_{\boldsymbol{w}}] = \boldsymbol{w}'\boldsymbol{\mu}$. While its variance is $\sigma^2[r_{\boldsymbol{w}}] = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$, its standard deviation is $\sigma[r_{\boldsymbol{w}}] = \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}$.

2.2. The mean-variance frontier

A portfolio is on the *mean-variance frontier* if there is no portfolio with the same mean and a smaller variance. Merton (1972) provides an analytical characterization of the portfolios on the mean-variance frontier. Some notation is useful to describe his characterization. Let $A \equiv \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_N$, $B \equiv \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $C \equiv \mathbf{1}_N'\boldsymbol{\Sigma}^{-1}\mathbf{1}_N$, and $D \equiv BC - A^2$ denote constants with B, C, and D being positive. Suppose that $A \neq 0$. Let $\boldsymbol{w}_0 \equiv \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}_N}{C}$ and $\boldsymbol{w}_1 \equiv \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{A}$ denote two portfolios on the mean-variance frontier. While $\boldsymbol{w}_0$ is the global minimum-variance portfolio and has an expected return of A/C, $\boldsymbol{w}_1$ has an expected return of B/A.

The portfolio on the mean-variance frontier with an expected return of $E \in \mathbb{R}$ is:

$$\boldsymbol{w}_E \equiv \theta_E \boldsymbol{w}_0 + (1 - \theta_E)\boldsymbol{w}_1 \tag{1}$$

where $\theta_E \equiv \frac{E - B/A}{A/C - B/A}$. For any portfolio $\boldsymbol{w}$ on the frontier, the following holds:

$$\frac{\sigma^2[r_{\boldsymbol{w}}]}{1/C} - \frac{(E[r_{\boldsymbol{w}}] - A/C)^2}{D/C^2} = 1. \tag{2}$$

As Fig. 1(a) illustrates, portfolios on the frontier are represented in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space by a hyperbola.[15] The dot ('•') plots $\boldsymbol{w}_0$, which has an expected return of A/C as noted earlier and a standard deviation of $\sqrt{1/C}$. The top and bottom half-lines show the asymptotes of the frontier: $E[r_{\boldsymbol{w}}] = A/C \pm \sqrt{D/C}\sigma[r_{\boldsymbol{w}}]$. As Fig. 1(b) illustrates, portfolios on the frontier

[14] The assumptions that short selling is allowed and asset weights are unbounded follow Merton (1972).

[15] While in general A can be negative, zero, or positive, Fig. 1(a) assumes that A is positive. Since A is assumed to be positive in this figure and C is always positive, A/C is positive in the figure. Similar observations apply to subsequent figures that use the value of A.

are represented in $(E[r_{\boldsymbol{w}}], \sigma^2[r_{\boldsymbol{w}}])$ space by a parabola. The leftmost and rightmost dots plot, respectively, $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$. While their respective expected returns are A/C and B/A as noted earlier, their respective variances are $1/C$ and B/A^2. Note that $\boldsymbol{w}_1$ is located at the point where a ray from the origin that goes through $\boldsymbol{w}_0$ crosses the frontier; see Roll (1992).

2.3. The optimal portfolio

Consider an investor with an exogenously given amount of wealth. The investor's preferences over portfolios are represented by the function $U : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by:

$$U(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}]) = E[r_{\boldsymbol{w}}] - (\gamma/2)\sigma^2[r_{\boldsymbol{w}}] \tag{3}$$

where $\gamma > 0$ is the investor's *risk aversion coefficient.*[16] Focusing on portfolios with the same variance, the investor's satisfaction is higher for portfolios with *higher* means. Focusing on portfolios with the same mean, the investor's satisfaction is higher for portfolios with *lower* variances since $\gamma > 0$. Noting that $U(E[r_{\boldsymbol{w}}]-(\gamma/2)\sigma^2[r_{\boldsymbol{w}}], 0) = U(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$, the *certainty-equivalent return* (CER) of portfolio $\boldsymbol{w}$ is $U(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$.

The investor's optimal portfolio solves:

$$\max_{\boldsymbol{w} \in \mathbb{R}^N} \quad E[r_{\boldsymbol{w}}] - (\gamma/2)\sigma^2[r_{\boldsymbol{w}}] \tag{4}$$

$$s.t. \qquad \boldsymbol{w}'\mathbf{1}_N = 1. \tag{5}$$

Eqs. (4) and (5) imply that this portfolio maximizes the function defined by Eq. (3) subject to asset weights summing to one. Asset weights are constrained to sum to one because the investor's wealth is assumed to be fully invested.

The following result characterizes the composition, expected return, and standard deviation of the investor's optimal portfolio.

[16] The use of this function by expected utility maximizers leads to the selection of optimal portfolios if: (1) risky asset returns have a multivariate elliptical distribution (such as the Normal or t-distributions) with finite first and second moments; or (2) utility functions are quadratic; see Ingersoll (1987, Ch. 4) and Huang and Litzenberger (1988, Ch. 3). When neither (1) or (2) holds, expected utility is under certain conditions well approximated by such a function; see Markowitz (1959, 2010) and Levy and Markowitz (1979).

Theorem 1. *The optimal portfolio of an investor with a risk aversion coefficient of* $\gamma \in \mathbb{R}_{++}$ *is:*

$$\boldsymbol{w}_\gamma \equiv \theta_\gamma \boldsymbol{w}_0 + (1-\theta_\gamma)\boldsymbol{w}_1 \tag{6}$$

where $\theta_\gamma \equiv \frac{E_\gamma - B/A}{A/C - B/A}$, *its expected return is:*

$$E_\gamma \equiv A/C + (D/C)/\gamma, \tag{7}$$

and its standard deviation is:

$$\sigma_\gamma \equiv \sqrt{1/C + (D/C)/\gamma^2}. \tag{8}$$

Using Eqs. (1) and (6), the optimal portfolio of an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$, $\boldsymbol{w}_\gamma$, is on the mean-variance frontier. Eqs. (7) and (8) imply that $\boldsymbol{w}_\gamma$'s expected return and standard deviation, respectively, E_γ and σ_γ, depend on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the values of A, C, and D in the case of E_γ and through the values of C and D in the case of σ_γ) as well as γ. Given $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, note that E_γ and σ_γ *decrease* in γ since $D/C > 0$.[17]

The next two corollaries examine the location of an investor's optimal portfolio along the mean-variance frontier.

Corollary 1. *(i) The optimal portfolio of an investor with a risk aversion coefficient of* $\gamma \in \mathbb{R}_{++}$, $\boldsymbol{w}_\gamma$, *lies above the global minimum-variance portfolio,* $\boldsymbol{w}_0$, *in* $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ *space; (ii) The former portfolio converges to the latter as* γ *converges to infinity.*

Corollary 1(i) can be seen as follows. Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$. Since $D/C > 0$ and $\gamma > 0$, Eq. (7) implies that the investor's optimal portfolio, $\boldsymbol{w}_\gamma$, has an expected return, E_γ, that exceeds A/C. Recall that the global minimum-variance portfolio, $\boldsymbol{w}_0$, has an expected return of A/C. Hence, $\boldsymbol{w}_\gamma$ lies above $\boldsymbol{w}_0$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space.

The leftmost and rightmost dots ('•') in Fig. 2(a) illustrate the location of, respectively,

[17] Here, only partial equilibrium results are discussed. Black (1972) characterizes expected asset returns in equilibria for economies where investors use Markowitz's model for portfolio selection.

$\boldsymbol{w}_0$ and $\boldsymbol{w}_\gamma$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. By definition, $\boldsymbol{w}_0$ lies at the leftmost point on the curve representing portfolios on the mean-variance frontier. The two thin dashed indifference curves are associated with γ; see Eq. (3). Note that $\boldsymbol{w}_\gamma$ lies at the point where the top thin dashed curve is tangent to the top half of the frontier. Hence, $\boldsymbol{w}_\gamma$ lies above $\boldsymbol{w}_0$.

Corollary 1(ii) can be seen as follows. Using Eq. (7), note that E_γ converges to A/C and thus θ_γ converges to one as γ converges to infinity. Hence, Eq. (6) implies that $\boldsymbol{w}_\gamma$ moves down along the top half of the mean-variance frontier toward $\boldsymbol{w}_0$ as γ converges to infinity.

Corollary 2. *The expected return of the optimal portfolio of an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$, E_γ, converges to infinity as γ converges to zero.*

Corollary 2 can be seen as follows. Since $D/C > 0$, Eq. (7) implies that E_γ converges to infinity as γ converges to zero. Hence, $\boldsymbol{w}_\gamma$ moves up unboundedly along the top half of the mean-variance frontier as γ converges to zero.

2.4. Loss in CER arising from the misspecification of the risk aversion coefficient

The loss in CER of an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$ who uses an erroneous risk aversion coefficient of $\gamma_\varepsilon \in \mathbb{R}_{++} \backslash \{\gamma\}$ in seeking to find the optimal portfolio is:

$$L_{\gamma,\gamma_\varepsilon} \equiv U(E[r_{\boldsymbol{w}_\gamma}], \sigma[r_{\boldsymbol{w}_\gamma}]) - U(E[r_{\boldsymbol{w}_{\gamma_\varepsilon}}], \sigma[r_{\boldsymbol{w}_{\gamma_\varepsilon}}]). \tag{9}$$

Fig. 2(a) illustrates $L_{\gamma,\gamma_\varepsilon}$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. As noted earlier, the rightmost dot ('•') shows that the investor's optimal portfolio, $\boldsymbol{w}_\gamma$, lies at the point where the top thin dashed indifference curve associated with γ is tangent to the top half of the mean-variance frontier. The middle dot shows that the portfolio that the investor selects when using an erroneous risk aversion coefficient of $\gamma_\varepsilon > \gamma$, $\boldsymbol{w}_{\gamma_\varepsilon}$, lies at the point where the thick dashed indifference curve associated with γ_ε is tangent to the top half of the mean-variance frontier. Since $\gamma_\varepsilon > \gamma$,

$\boldsymbol{w}_{\gamma_\varepsilon}$ lies below $\boldsymbol{w}_\gamma$.[18] The bottom thin dashed indifference curve is associated to γ and goes through $\boldsymbol{w}_{\gamma_\varepsilon}$. Note that $L_{\gamma,\gamma_\varepsilon}$ is the vertical distance between the top and bottom thin dashed curves.

The following result provides an analytical expression for the loss in CER.

Theorem 2. *An investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$ who uses an erroneous risk aversion coefficient of $\gamma_\varepsilon \in \mathbb{R}_{++}\backslash\{\gamma\}$ in seeking to find the optimal portfolio has a loss in CER of:*

$$L_{\gamma,\gamma_\varepsilon} = (D/C)\,(\gamma_\varepsilon - \gamma)^2/(2\gamma\gamma_\varepsilon^2). \tag{10}$$

Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$ who uses an erroneous risk aversion coefficient of $\gamma_\varepsilon \in \mathbb{R}_{++}\backslash\{\gamma\}$ in seeking to find the optimal portfolio. Two observations follow from Eq. (10). First, since D/C, $|\gamma_\varepsilon - \gamma|$, γ, and γ_ε are positive, so is $L_{\gamma,\gamma_\varepsilon}$. Second, $L_{\gamma,\gamma_\varepsilon}$ depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the term D/C) as well as γ and γ_ε.

Given $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and $\gamma \in \mathbb{R}_{++}$, Fig. 2(b) illustrates how $L_{\gamma,\gamma_\varepsilon}$ depends on γ_ε; see Eq. (10). If $0 < \gamma_\varepsilon < \gamma$, then $L_{\gamma,\gamma_\varepsilon}$: (a) decreases in γ_ε; (b) is unbounded from above since $L_{\gamma,\gamma_\varepsilon}$ converges to infinity as γ_ε converges to zero; and (c) is strictly convex on γ_ε.[19] If $\gamma_\varepsilon > \gamma$, then $L_{\gamma,\gamma_\varepsilon}$: (a) increases in γ_ε; (b) is bounded from above since $L_{\gamma,\gamma_\varepsilon}$ converges to $(D/C)\,/(2\gamma)$ as γ_ε converges to infinity; (c) is strictly convex on γ_ε if $\gamma < \gamma_\varepsilon < 3\gamma/2$;[20] and (d) is strictly concave on γ_ε if $\gamma_\varepsilon > 3\gamma/2$. Importantly, the value of $L_{\gamma,\gamma_\varepsilon}$ if γ_ε is less than γ by an amount strictly between zero and γ *exceeds* the value of $L_{\gamma,\gamma_\varepsilon}$ if γ_ε is greater than γ by the same amount. For example, $L_{\gamma,\gamma_\varepsilon} = (D/C)\,/(2\gamma)$ if $\gamma_\varepsilon = \gamma/2$ but $L_{\gamma,\gamma_\varepsilon} = (D/C)\,/(18\gamma)$ if $\gamma_\varepsilon = 3\gamma/2$.

The intuition for why the loss in CER if γ_ε is less than γ by an amount strictly between

[18] Similar graphical results hold in the case where $0 < \gamma_\varepsilon < \gamma$. In this case, however, $\boldsymbol{w}_{\gamma_\varepsilon}$ lies above $\boldsymbol{w}_\gamma$.

[19] Note that $\partial^2 L_{\gamma,\gamma_\varepsilon}/\partial\gamma_\varepsilon^2 = (D/C)(3\gamma - 2\gamma_\varepsilon)/\gamma_\varepsilon^4$; see Eq. (10).

[20] Since $\partial^2 L_{\gamma,\gamma_\varepsilon}/\partial\gamma_\varepsilon^2 = (D/C)(3\gamma - 2\gamma_\varepsilon)/\gamma_\varepsilon^4$ and $D/C > 0$, the inflection point of $L_{\gamma,\gamma_\varepsilon}$ occurs when $\gamma_\varepsilon = 3\gamma/2$ and $L_{\gamma,\gamma_\varepsilon} = (D/C)/(18\gamma)$; see Eq. (10).

zero and γ *exceeds* the loss in CER if γ_ε is greater than γ by the same amount is as follows. Since the *difference* between the expected returns of the optimal and global-minimum variance portfolios is *inversely* proportional to γ, the use of the former value of γ_ε leads to the selection of a portfolio on the mean-variance frontier with an expected return that *deviates more* from that of the optimal portfolio than the use of the latter; see Eqs. (6) and (7).[21] Hence, the use of the value of γ_ε less than γ results in a *larger* loss in CER than the use of the value of γ_ε greater than γ.[22]

The next result examines the losses in CER when γ_ε is either less than or greater than γ by some percentage.

Theorem 3. *Consider an investor with a risk aversion coefficient of* $\gamma \in \mathbb{R}_{++}$ *who uses an erroneous risk aversion coefficient of* $\gamma_\varepsilon \in \mathbb{R}_{++} \backslash \{\gamma\}$ *in seeking to find the optimal portfolio.*

(i) If $\gamma_\varepsilon = \gamma_{\varepsilon,\kappa^-} \equiv (1-\kappa)\gamma$ *where* $\kappa \in (0,1)$, *then the loss in CER is:*

$$L_{\gamma,\gamma_{\varepsilon,\kappa^-}} = \left[(D/C)/(2\gamma)\right]\left[\kappa/(1-\kappa)\right]^2. \tag{11}$$

(ii) If $\gamma_\varepsilon = \gamma_{\varepsilon,\kappa^+} \equiv (1+\kappa)\gamma$ *where* $\kappa \in (0,\infty)$, *then the loss in CER is:*

$$L_{\gamma,\gamma_{\varepsilon,\kappa^+}} = \left[(D/C)/(2\gamma)\right]\left[\kappa/(1+\kappa)\right]^2. \tag{12}$$

(iii) If $\kappa \in (0,1)$, *then:*

$$(L_{\gamma,\gamma_{\varepsilon,\kappa^-}}/L_{\gamma,\gamma_{\varepsilon,\kappa^+}}) - 1 = RCL_\kappa \equiv \left[(1+\kappa)/(1-\kappa)\right]^2 - 1. \tag{13}$$

Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$. In seeking to find the optimal portfolio, suppose that the investor uses an erroneous risk aversion coefficient *less* than γ by some percentage $\kappa \in (0,1)$, $\gamma_{\varepsilon,\kappa^-}$. Using Eq. (11), the loss in CER, $L_{\gamma,\gamma_{\varepsilon,\kappa^-}}$, depends

[21] For example, while the use of $\gamma_\varepsilon = \gamma/2$ leads to the selection of a portfolio with an expected return *greater* than that of the optimal portfolio by $(D/C)/\gamma$, the use of $\gamma_\varepsilon = 3\gamma/2$ leads to the selection of a portfolio with an expected return *less* than that of the optimal portfolio by $(1/3)(D/C)/\gamma$.

[22] The CER of a portfolio on the mean-variance frontier can be seen of as a function of the expected return of the portfolio. Since this function is *symmetric* around the expected return of the optimal portfolio, the use of the value of γ_ε less than γ and the corresponding *larger* deviation from the expected return of the optimal portfolio result in a *larger* loss in CER than the use of the value of γ_ε greater than γ.

on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the term D/C) as well as γ and κ. Given $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and $\gamma \in \mathbb{R}_{++}$, note that $L_{\gamma,\gamma_{\varepsilon,\kappa^-}}$ increases in κ. Given $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and $\kappa \in (0,1)$, note that $L_{\gamma,\gamma_{\varepsilon,\kappa^-}}$ decreases in γ.[23]

In seeking to find the optimal portfolio, suppose now that the investor uses an erroneous risk aversion coefficient *greater* than γ by some percentage $\kappa \in (0,\infty)$, $\gamma_{\varepsilon,\kappa^+}$. The results for the loss in CER, $L_{\gamma,\gamma_{\varepsilon,\kappa^+}}$, are similar to the results noted for $L_{\gamma,\gamma_{\varepsilon,\kappa^-}}$; see Eq. (12). However, for any $\kappa \in (0,1)$, Eqs. (11) and (12) imply that $L_{\gamma,\gamma_{\varepsilon,\kappa^-}} > L_{\gamma,\gamma_{\varepsilon,\kappa^+}}$. Hence, the loss in CER when γ_ε is less than γ by some percentage κ strictly between zero and one *exceeds* the loss in CER when γ_ε is greater than γ by the same percentage.

Using Eq. (13), the relative change in the loss in CER arising from an investor using $\gamma_{\varepsilon,\kappa^-}$ instead of $\gamma_{\varepsilon,\kappa^+}$ in seeking to find the optimal portfolio, RCL_κ: (i) is positive (consistent with the fact that $L_{\gamma,\gamma_{\varepsilon,\kappa^-}} > L_{\gamma,\gamma_{\varepsilon,\kappa^+}}$); (ii) increases in κ; and (iii) does not depend on $\boldsymbol{\mu}$ or $\boldsymbol{\Sigma}$ or γ. Fig. 2(c) reports the values of RCL_κ when κ ranges from zero to 30%. Since these values of RCL_κ are sizeable except if κ is zero or very close to zero, a key practical implication follows. In attempting to infer the preferences of an investor to implement Markowitz's model, practitioners should be aware that the loss in CER if the erroneous risk aversion coefficient is less than the 'true' risk aversion coefficient by some amount can notably exceed the loss in CER if the former coefficient is greater than the latter by the same amount.[24]

[23] Das et al. (2010) assess the loss in CER in a numerical example. Fixing the value of κ, they find that the *average* loss in CER across two values of γ_ε given by $\gamma_{\varepsilon,\kappa^-}$ and $\gamma_{\varepsilon,\kappa^+}$ is larger for smaller values of γ. However, they do not provide an analytical expression for the loss in CER nor the average loss in CER. Using Eqs. (11) and (12), the average loss in CER is $[(D/C)/(2\gamma)]\,[\kappa^2(1+\kappa^2)]/[(1-\kappa^2)^2]$.

[24] For any risk aversion coefficient $\gamma \in \mathbb{R}_{++}$, let $t = 1/\gamma$ denote the corresponding *risk tolerance coefficient*. Let L_{t,t_ε} denote the loss in CER of an investor with a 'true' risk tolerance coefficient of $t \in \mathbb{R}_{++}$ who uses an erroneous risk tolerance coefficient of $t_\varepsilon \in \mathbb{R}_{++}\backslash\{t\}$ in seeking to find the optimal portfolio. Eq. (10) implies that $L_{t,t_\varepsilon} = (D/C)\,t(1-t_\varepsilon/t)^2/2$. Letting $t_{\varepsilon,l^-} \equiv (1-l)t$ and $t_{\varepsilon,l^+} \equiv (1+l)t$ where $l \in (0,1)$, it can be seen that $L_{t,t_{\varepsilon,l^-}} = L_{t,t_{\varepsilon,l^+}} = (D/C)\,tl^2/2$. Hence, the loss in CER arising from using an erroneous risk tolerance coefficient less than the 'true' risk tolerance coefficient by some amount *equals* the loss in CER arising from using an erroneous risk tolerance coefficient greater than the 'true' risk tolerance coefficient by the same amount. Since Das et al. (2010) parameterize the loss in CER by the erroneous risk aversion coefficient (*not* by the erroneous risk tolerance coefficient), so does Section 2.4.

3. The model of Das et al. (2010)

This section examines the model of Das et al. (2010).

3.1. Assumptions

In addition to the assumptions in Section 2.1 on assets and portfolios, suppose that asset returns have a multivariate Normal distribution.[25]

3.2. Optimal portfolios within accounts

Consider an investor with an exogenously given amount of wealth. This amount of wealth is exogenously allocated to an exogenously given number of accounts denoted by $M > 1$. Letting $\mathbf{1}_M$ denote the $M \times 1$ unit vector, the $M \times 1$ vector of fractions of the investor's wealth in the accounts is $\boldsymbol{y} \in \mathbb{R}^M_{++}$ where $\boldsymbol{y}'\mathbf{1}_M = 1$. The mth entry of $\boldsymbol{y}$ is the fraction of the investor's wealth in account m. The investor allocates the wealth within each account among the *same* set of assets that are available for trade. However, the fraction of wealth in an account that the investor allocates to a given asset possibly *depends* on the account.

The optimal portfolio within account a given account $m \in \{1, ..., M\}$ solves:

$$\max_{\boldsymbol{w} \in \mathbb{R}^N} \quad E[r_{\boldsymbol{w}}] \tag{14}$$

$$s.t. \qquad \boldsymbol{w}'\mathbf{1}_N = 1 \tag{15}$$

$$P[r_{\boldsymbol{w}} \leq H_m] \leq \alpha_m \tag{16}$$

where $P[\,\cdot\,]$ denotes probability, $H_m \in \mathbb{R}$ is account m's threshold return, and $\alpha_m \in (0, 1/2)$ is account m's threshold probability. Hence, this portfolio maximizes account m's expected return subject to: (i) fully investing the wealth in the account; and (ii) the probability of

[25] The results in Das et al.'s model hold more generally in the case where asset returns are assumed to have a multivariate elliptical distribution with finite first and second moments. Moreover, these results hold as an approximation if the multivariate distribution of asset returns is unknown but has finite first and second moments; see O'cinneide (1990). Das and Statman (2013) examine the composition of optimal portfolios within accounts when asset returns are assumed to have non-elliptical distributions. Additionally, the results in Das et al.'s model extend to the case where a risk-free asset is assumed to be available for trade; see Alexander et al. (2020) and references therein.

the account's return being less than or equal to H_m not exceeding α_m. Note that probability constraint (16) loosens if α_m increases but tightens if H_m increases. Therefore, the size of thresholds α_m and H_m reflects the investor's goal for account m.

Probability constraint (16) can be seen as a restriction on account m's tail risk. Some notation is useful to formally write such a restriction. For any $\alpha \in (0, 1/2)$, let $z_\alpha \equiv -\Phi^{-1}(\alpha)$ where $\Phi(\cdot)$ denotes the cumulative univariate standard Normal distribution function. Since $\alpha \in (0, 1/2)$, note that z_α is positive and decreases in α.

The Value-at-Risk (VaR) at confidence level $1 - \alpha$ of portfolio $\boldsymbol{w}$ is:[26]

$$V[1-\alpha, r_{\boldsymbol{w}}] \equiv z_\alpha \sigma[r_{\boldsymbol{w}}] - E[r_{\boldsymbol{w}}]. \tag{17}$$

Note that portfolio $\boldsymbol{w}$ meets probability constraint (16) if and only if:

$$V[1-\alpha_m, r_{\boldsymbol{w}}] \leq -H_m. \tag{18}$$

Hence, probability constraint (16) restricts account m's VaR at confidence level $1 - \alpha_m$ to be $-H_m$ or less. Using Eqs. (17) and (18), probability constraint (16) is equivalent to:

$$E[r_{\boldsymbol{w}}] \geq H_m + z_{\alpha_m} \sigma[r_{\boldsymbol{w}}]. \tag{19}$$

Hence, as Fig. 3(a) illustrates, portfolios that lie on or above a line with an intercept of H_m and a slope of z_{α_m} in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space meet probability constraint (16), whereas portfolios that lie below the line do not meet the constraint.[27] Given a threshold probability of α_m, Fig. 3(b) shows that increasing the threshold return from H_m^{low} to H_m^{high} tightens the constraint because the intercept of the line associated with the constraint also increases from H_m^{low} to H_m^{high}. Given a threshold return of H_m, Fig. 3(c) shows that increasing the threshold probability from α_m^{low} to α_m^{high} loosens the constraint because the slope of the line associated

[26] Like a portfolio's VaR, a portfolio's Conditional Value-at-Risk (CVaR) is a linear function of its mean and standard deviation if asset returns have a multivariate Normal distribution. Here, a portfolio's CVaR at a given confidence level is the portfolio's expected loss given that the loss equals or exceeds the portfolio's VaR at that confidence level. The results in Das et al.'s model extend to the case where CVaR is used instead of VaR to set account goals. Hull (2023, Ch. 11) compares the theoretical properties of VaR and CVaR.

[27] While threshold returns can in general be negative, zero, or positive, Fig. 3(a) assumes that $H_m < 0$. Subsequent figures that use threshold returns similarly make an assumption on the sign of threshold returns.

with the constraint decreases from $z_{\alpha_m^{low}}$ to $z_{\alpha_m^{high}}$ (recall that z_α decreases in α).

Optimal portfolios within accounts may or may not exist depending on the thresholds and optimization inputs. In order to identify conditions on the thresholds and optimization inputs under which these portfolios exist, a characterization of the existence of the global minimum-VaR portfolio and its VaR if it exists is useful. The following notation is used to describe such a characterization. Let:

$$\overline{\alpha} \equiv \Phi(-\sqrt{D/C}). \tag{20}$$

Using Eq. (20), note that $\overline{\alpha}$ depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the values of C and D). Since $-\sqrt{D/C} < 0$, Eq. (20) implies that $\overline{\alpha} \in (0, 1/2)$. For any $\alpha \in (0, \overline{\alpha})$, let:

$$\overline{H}_\alpha \equiv A/C - \sqrt{(z_\alpha^2 - D/C)/C}. \tag{21}$$

Using Eq. (21), note that $\overline{H}_\alpha$ depends on α (through the value of z_α) as well as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the values of A, C, and D).

Alexander and Baptista (2002) characterize the existence of the global minimum-VaR portfolio as well as its composition, expected return, standard deviation, and VaR if it exists.

Theorem 4. *(i) The global minimum-VaR portfolio at confidence level $1-\alpha$ exists if and only if $\alpha \in (0, \overline{\alpha})$; (ii) If $\alpha \in (0, \overline{\alpha})$, then this portfolio is:*

$$\boldsymbol{w}_{1-\alpha} \equiv \theta_{1-\alpha}\boldsymbol{w}_0 + (1-\theta_{1-\alpha})\boldsymbol{w}_1 \tag{22}$$

where $\theta_{1-\alpha} \equiv \frac{E_{1-\alpha}-B/A}{A/C-B/A}$, its expected return is:

$$E_{1-\alpha} \equiv A/C + \sqrt{(D^2/C^3)\,/\,(z_\alpha^2 - D/C)}, \tag{23}$$

its standard deviation is:

$$\sigma_{1-\alpha} \equiv \sqrt{(z_\alpha^2/C)\,/\,(z_\alpha^2 - D/C)}, \tag{24}$$

and its VaR at confidence level $1-\alpha$ is:

$$V_{1-\alpha} \equiv -\overline{H}_\alpha. \tag{25}$$

Using Theorem 4(i), the existence of the global minimum-VaR portfolio depends on the confidence level $1-\alpha$ as well as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (since $\overline{\alpha}$ depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$). First, if $1-\alpha$ is less than or equal to $1-\overline{\alpha}$, then the portfolio does not exist. Its non-existence can be seen by noting that the slope of the representation of the portfolios on the top half of the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space exceeds z_α if $1-\alpha \leq 1-\overline{\alpha}$ (or, equivalently, $\alpha \geq \overline{\alpha}$).[28] Using Eq. (17), the VaRs of such portfolios decrease when moving up along the frontier. Hence, the problem of globally minimizing VaR does not have a solution.

Second, if $1-\alpha$ is greater than $1-\overline{\alpha}$, then the global minimum-VaR portfolio, $\boldsymbol{w}_{1-\alpha}$, exists. Eqs. (1) and (22) imply that $\boldsymbol{w}_{1-\alpha}$ is on the mean-variance frontier. The leftmost and rightmost dots ('•') in Fig. 4 illustrate the location of, respectively, $\boldsymbol{w}_0$ and $\boldsymbol{w}_{1-\alpha}$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. While $\boldsymbol{w}_{1-\alpha}$ lies above $\boldsymbol{w}_0$, $\boldsymbol{w}_{1-\alpha}$ converges to $\boldsymbol{w}_0$ as $1-\alpha$ converges to 100% from below (or, equivalently, α converges to zero from above); see Eq. (23).[29] Also, $\boldsymbol{w}_{1-\alpha}$ lies at the point where a line with a slope of z_α is tangent to the top half of the mean-variance frontier. The intercept of this line is $\overline{H}_\alpha = -V_{1-\alpha}$; see Eq. (25).

The following result characterizes the existence of the optimal portfolio within a given account as well as its composition, expected return, standard deviation, and VaR if it exists.

Theorem 5. *Consider an account $m \in \{1, ..., M\}$ with threshold probability α_m and threshold return H_m. (i) The optimal portfolio within account m exists if and only if $\alpha_m \in (0, \overline{\alpha})$ and $H_m \in (-\infty, \overline{H}_{\alpha_m}]$;*[30] *(ii) If $\alpha_m \in (0, \overline{\alpha})$ and $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, then this portfolio is:*

$$\boldsymbol{w}_{\alpha_m, H_m} \equiv \theta_{\alpha_m, H_m} \boldsymbol{w}_0 + (1-\theta_{\alpha_m, H_m}) \boldsymbol{w}_1 \tag{26}$$

[28] While the slope of the representation of the portfolios on the top half of the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space exceeds $\sqrt{D/C}$, z_α is less than or equal to $\sqrt{D/C}$ if $\alpha \geq \overline{\alpha}$; see Eqs. (2) and (20).

[29] Note that z_α and $E_{1-\alpha}$ converge to, respectively, infinity and A/C as α converges to zero from above.

[30] The existence of optimal portfolios within accounts is closely related to Theorem 4. The global minimum-VaR portfolio at the confidence level of $1-\alpha_m$ exists if and only if $\alpha_m \in (0, \overline{\alpha})$. Moreover, when $\alpha_m \in (0, \overline{\alpha})$, there is a portfolio meeting probability constraint (16) if and only if $(-\infty, \overline{H}_{\alpha_m}]$ where $\overline{H}_{\alpha_m}$ equals minus one multiplied by the VaR of the global minimum-VaR portfolio at the confidence level of $1-\alpha_m$.

where $\theta_{\alpha_m,H_m} \equiv \frac{E_{\alpha_m,H_m}-B/A}{A/C-B/A}$, *its expected return is:*

$$E_{\alpha_m,H_m} \equiv A/C + \sqrt{(D/C)(\sigma^2_{\alpha_m,H_m} - 1/C)}, \tag{27}$$

its standard deviation is:

$$\sigma_{\alpha_m,H_m} \equiv \frac{z_{\alpha_m}(A/C - H_m) + \sqrt{(D/C)[(A/C - H_m)^2 - (z^2_{\alpha_m} - D/C)/C]}}{z^2_{\alpha_m} - D/C}, \tag{28}$$

and its VaR at confidence level $1 - \alpha_m$ *is:*

$$V_{\alpha_m,H_m} \equiv -H_m. \tag{29}$$

Consider an account $m \in \{1, ..., M\}$ with threshold probability α_m and threshold return H_m. Using Theorem 5(i), the optimal portfolio within account m exists if and only if $\alpha_m \in (0, \overline{\alpha})$ and $H_m \in (-\infty, \overline{H}_{\alpha_m}]$.[31] Hence, its existence depends on α_m and H_m as well as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (since $\overline{\alpha}$ and $\overline{H}_{\alpha_m}$ depend on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$). Its non-existence occurs in two cases. First, if $\alpha_m \in [\overline{\alpha}, 1/2)$, then probability constraint (16) is overly *loose* regardless of the value of H_m. As Fig. 5(a) illustrates, the optimal portfolio within account m does not exist in this case because the set of expected returns of portfolios that meet this constraint does not have a finite upper bound. While the curve represents portfolios on the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space, the line has an intercept of H_m and a slope of z_{α_m}. Since portfolios on the frontier with sufficiently large expected returns lie above the line and thus meet probability constraint (16), the set of expected returns of portfolios that meet this constraint does not have a finite upper bound. Noting that the optimal portfolio within account m would have the maximum expected return among the portfolios that meet such a constraint, the former portfolio does not exist; see Eqs. (14)–(16).

Second, if $\alpha_m \in (0, \overline{\alpha})$ and $H_m \in (\overline{H}_{\alpha_m}, \infty)$, then probability constraint (16) is overly

[31] While Das et al. (2010, pp. 325 and 326) examine the problem of existence of the optimal portfolio within a given account, they do not analytically identify the thresholds for which this portfolio exists. Specifically, given the threshold probability of the account, they formulate the problem of finding the maximum threshold return for which such a portfolio exists and note that this problem can be numerically solved.

tight. As Fig. 5(b) illustrates, the optimal portfolio within account m does not exist in this case because no portfolio meets probability constraint (16). By definition, all portfolios lie either on or to the right of the curve that represents portfolios on the mean-variance frontier. Any portfolio that would meet probability constraint (16) would lie on or above a line with an intercept of H_m and a slope of z_{α_m}. Since the line is above the curve, no portfolio meets probability constraint (16). Hence, the optimal portfolio within account m does not exist.

If $\alpha_m \in (0, \overline{\alpha})$ and $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, then the optimal portfolio within account m, $\boldsymbol{w}_{\alpha_m,H_m}$, exists and is on the mean-variance frontier;[32] see Theorem 5(ii) along with Eqs. (1) and (26). Its expected return and standard deviation, respectively, E_{α_m,H_m} and σ_{α_m,H_m}, depend on α_m and H_m as well as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the values of A, C, and D); see Eqs. (27) and (28). Given $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and $\alpha_m \in (0, \overline{\alpha})$, note that E_{α_m,H_m} and σ_{α_m,H_m} *decrease* in H_m if $H_m \in (-\infty, \overline{H}_{\alpha_m})$. Given $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, note that E_{α_m,H_m} and σ_{α_m,H_m} *increase* in α_m if $\alpha_m \in (0, \overline{\alpha})$. Its VaR at confidence level $1 - \alpha_m$, V_{α_m,H_m}, is $-H_m$; see Eq. (29).

The next four corollaries examine the location of optimal portfolios within accounts along the mean-variance frontier.

Corollary 3. *Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$. (i) If $H_m \in (-\infty, \overline{H}_{\alpha_m})$, then the optimal portfolio within account m, $\boldsymbol{w}_{\alpha_m,H_m}$, lies above the global minimum-VaR portfolio at confidence level $1 - \alpha_m$, $\boldsymbol{w}_{1-\alpha_m}$, in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space; (ii) If $H_m = \overline{H}_{\alpha_m}$, then the former portfolio equals the latter.*

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$. Assuming that $H_m < \overline{H}_{\alpha_m}$, Fig. 6(a) illustrates Corollary 3(i). The

[32] Das et al. derive a semi-analytical expression for $\boldsymbol{w}_{\alpha_m,H_m}$ that requires a numerical approach.

curve represents portfolios on the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space, whereas the line has an intercept of H_m and a slope of z_{α_m}. The rightmost dot ('•') shows that $\boldsymbol{w}_{\alpha_m,H_m}$ lies at the point where the line crosses the top half of the frontier. The leftmost dot shows the global minimum-VaR portfolio at confidence level $1-\alpha_m$, $\boldsymbol{w}_{1-\alpha_m}$. Note that $\boldsymbol{w}_{\alpha_m,H_m}$ lies above $\boldsymbol{w}_{1-\alpha_m}$. Assuming that $H_m = \overline{H}_{\alpha_m}$, Fig. 6(b) illustrates Corollary 3(ii). The dot shows that $\boldsymbol{w}_{\alpha_m,H_m}$ lies at the point where the line is tangent to the top half of the frontier. Hence, $\boldsymbol{w}_{\alpha_m,H_m}$ equals $\boldsymbol{w}_{1-\alpha_m}$.

Corollary 4. *Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$. The expected return of the optimal portfolio within account m, E_{α_m,H_m}, converges to infinity as H_m converges to minus infinity.*

Corollary 4 can be seen as follows. Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$. Since $z^2_{\alpha_m} > D/C > 0$, Eq. (28) implies that the standard deviation of the optimal portfolio within account m, σ_{α_m,H_m}, converges to infinity as H_m converges to minus infinity. It follows from Eq. (27) that the expected return of this portfolio, E_{α_m,H_m}, also converges to infinity as H_m converges to minus infinity.

For any threshold return $H_m \in (-\infty, A/C)$, let:

$$\underline{\alpha}_{H_m} \equiv \Phi\left(-\sqrt{D/C + C(A/C - H_m)^2}\right). \tag{30}$$

Since C and D are positive, Eqs. (20) and (30) imply that $\underline{\alpha}_{H_m} \in (0, \overline{\alpha})$.

Corollary 5. *Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in [\underline{\alpha}_{H_m}, \overline{\alpha})$ and threshold return $H_m \in (-\infty, A/C)$. (i) If $\alpha_m \in (\underline{\alpha}_{H_m}, \overline{\alpha})$, then the optimal portfolio within account m, $\boldsymbol{w}_{\alpha_m,H_m}$, lies above the global minimum-VaR portfolio at confidence level $1-\alpha_m$, $\boldsymbol{w}_{1-\alpha_m}$, in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space; (ii) If $\alpha_m = \underline{\alpha}_{H_m}$, then the former portfolio equals* *the latter.*

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in [\underline{\alpha}_{H_m}, \overline{\alpha})$ and threshold return $H_m \in (-\infty, A/C)$. Assuming that $\alpha_m \in (\underline{\alpha}_{H_m}, \overline{\alpha})$, Fig. 6(c) illustrates Corollary 5(i). The rightmost dot ('•') shows that $\boldsymbol{w}_{\alpha_m,H_m}$ lies in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space at the point where a line with an intercept of H_m and a slope of z_{α_m} crosses the top half of the frontier. The leftmost dot plots the global minimum-VaR portfolio at confidence level $1 - \underline{\alpha}_{H_m}$, $\boldsymbol{w}_{1-\underline{\alpha}_{H_m}}$. Hence, $\boldsymbol{w}_{\alpha_m,H_m}$ lies above $\boldsymbol{w}_{1-\underline{\alpha}_{H_m}}$. Assuming that $\alpha_m = \underline{\alpha}_{H_m}$, Fig. 6(d) illustrates Corollary 5(ii). The dot shows that $\boldsymbol{w}_{\alpha_m,H_m}$ lies at the point where the line is tangent to the top half of the frontier. Hence, $\boldsymbol{w}_{\alpha_m,H_m}$ equals $\boldsymbol{w}_{1-\underline{\alpha}_{H_m}}$.

Corollary 6. *Consider an account $m \in \{1, ..., M\}$ with threshold return $H_m \in (-\infty, A/C)$. The expected return of the optimal portfolio within account m, E_{α_m,H_m}, converges to infinity as the threshold probability α_m converges to $\overline{\alpha}$ from below.*

Corollary 6 can be seen as follows. Consider an account $m \in \{1, ..., M\}$ with threshold return $H_m \in (-\infty, A/C)$. Using the definition of z_{α_m} and Eq. (20), z_{α_m} converges to $\sqrt{D/C}$ as α_m converges to $\overline{\alpha}$ from below. Therefore, the assumption that $H_m < A/C$ and Eq. (28) imply that the standard deviation of the optimal portfolio within account m, σ_{α_m,H_m}, converges to infinity as α_m converges to $\overline{\alpha}$ from below. It follows from Eq. (27) that the expected return of this portfolio, E_{α_m,H_m}, also converges to infinity as α_m converges to $\overline{\alpha}$ from below.

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$, threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, and optimal portfolio $\boldsymbol{w}_{\alpha_m,H_m}$. It turns out that there are infinitely many pairs of thresholds that lead to the same optimal portfolio. Some notation is useful to identify such pairs. Let:[33]

$$\overline{\overline{\alpha}} \equiv \Phi\left(-\sqrt{(D/C)[\sigma^2_{\alpha_m,H_m}/(\sigma^2_{\alpha_m,H_m} - 1/C)]}\right). \quad (31)$$

Since $C > 0$, $D > 0$, and $\sigma^2_{\alpha_m,H_m} > 0$, Eqs. (20) and (31) imply that $\overline{\overline{\alpha}} \in (0, \overline{\alpha})$. For any $\alpha \in (0, \overline{\overline{\alpha}}]$, let:[34]

$$H_\alpha \equiv E_{\alpha_m,H_m} - z_\alpha \sigma_{\alpha_m,H_m}. \quad (32)$$

Since z_α decreases in α, $z_\alpha > 0$, and $\sigma_{\alpha_m,H_m} > 0$, Eq. (32) implies that H_α increases in α. Given an account for which the optimal portfolio exists, the following result identifies the pairs of thresholds for another account with the same optimal portfolio.

Theorem 6. *Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$. Another account with threshold probability $\alpha \in (0, \overline{\overline{\alpha}}]$ and threshold return H_α has the same optimal portfolio as account m.*

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$, threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, and optimal portfolio $\boldsymbol{w}_{\alpha_m,H_m}$. Consider another account with threshold probability $\alpha \in (0, \overline{\overline{\alpha}}]$ and threshold return H_α. As Fig. 7(a) illustrates, if $\alpha \in (0, \overline{\overline{\alpha}})$, then the optimal portfolio within this account is $\boldsymbol{w}_{\alpha_m,H_m}$ because it lies at the point where a line with an intercept of H_α and a slope of z_α crosses the top half of the curve representing portfolios on the mean-variance frontier. As Fig. 7(a) also illustrates, if $\alpha = \overline{\overline{\alpha}}$, then the optimal portfolio within such an account is again $\boldsymbol{w}_{\alpha_m,H_m}$ because it lies at the point where a line with an intercept of $H_{\overline{\overline{\alpha}}}$ and a slope of $z_{\overline{\overline{\alpha}}}$ is tangent to the top half of the curve. Hence, $\boldsymbol{w}_{\alpha_m,H_m}$ equals the global minimum-VaR portfolio at confidence level $1 - \overline{\overline{\alpha}}$, $\boldsymbol{w}_{1-\overline{\overline{\alpha}}}$.[35]

[33] While $\overline{\overline{\alpha}}$ depends on α_m and H_m, the notation '$\overline{\overline{\alpha}}$' (instead of, e.g., '$\overline{\overline{\alpha}}_{\alpha_m,H_m}$') is used for brevity.

[34] While H_α depends on α_m and H_m, the notation 'H_α' (instead of, e.g., 'H_{α,α_m,H_m}') is used for brevity.

[35] If $\alpha \in (\overline{\overline{\alpha}}, \overline{\alpha})$, then $\boldsymbol{w}_{\alpha,H_\alpha}$ lies on the top half of the mean-variance frontier at the same point as or above $\boldsymbol{w}_{1-\alpha}$, which in turn lies above $\boldsymbol{w}_{\alpha_m,H_m}$; see Corollary 3. Therefore, an account with a threshold probability of $\alpha \in (\overline{\overline{\alpha}}, \overline{\alpha})$ cannot have an optimal portfolio equal to $\boldsymbol{w}_{\alpha_m,H_m}$. Hence, Theorem 6 imposes the condition that $\alpha \in (0, \overline{\overline{\alpha}}]$.

Fig. 7(b) illustrates the pairs of thresholds $\{(\alpha, H_\alpha)\}_{\alpha\in(0,\overline{\overline{\alpha}}]}$ of an account with an optimal portfolio equal to $\boldsymbol{w}_{\alpha_m,H_m}$.[36] Note that H_α: (i) increases in α; (ii) converges to $H_{\overline{\overline{\alpha}}}$ as α converges to $\overline{\overline{\alpha}}$ from below; and (iii) converges to minus infinity as α converges to zero from above.

Since $\boldsymbol{w}_{\alpha_m,H_m}$ is on the mean-variance frontier, $\boldsymbol{w}_{\alpha_m,H_m}$ solves:

$$\max_{\boldsymbol{w}\in\mathbb{R}^N} \quad E[r_{\boldsymbol{w}}] - (\gamma^i_{\alpha_m,H_m}/2)\sigma^2[r_{\boldsymbol{w}}] \tag{33}$$

$$s.t. \quad \boldsymbol{w}'\mathbf{1}_N = 1 \tag{34}$$

for some $\gamma^i_{\alpha_m,H_m} > 0$. Hereafter, $\gamma^i_{\alpha_m,H_m}$ is referred to as the *risk aversion coefficient implied by the optimal portfolio within account* m. The following result characterizes this coefficient.

Theorem 7. *Consider an account* $m \in \{1, ..., M\}$ *with threshold probability* $\alpha_m \in (0, \overline{\alpha})$ *and threshold return* $H_m \in (-\infty, \overline{H}_{\alpha_m}]$. *The risk aversion coefficient implied by the optimal portfolio within account* m *is:*

$$\gamma^i_{\alpha_m,H_m} = (D/C) \,/\, (E_{\alpha_m,H_m} - A/C) \tag{35}$$

where E_{α_m,H_m} *is given by Eq. (27).*

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$, threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, and optimal portfolio $\boldsymbol{w}_{\alpha_m,H_m}$. Using Eqs. (27) and (35), the risk aversion coefficient implied by this portfolio, $\gamma^i_{\alpha_m,H_m}$, depends on α_m and H_m (through the term E_{α_m,H_m}) as well as on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the terms D/C, E_{α_m,H_m}, and A/C).[37] Given α_m, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$, recall that E_{α_m,H_m} decreases in H_m and thus $\gamma^i_{\alpha_m,H_m}$ increases in H_m. Given H_m, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$, recall that E_{α_m,H_m} increases in α_m and thus $\gamma^i_{\alpha_m,H_m}$ decreases in α_m.

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$. Of interest

[36] Given the thresholds of an account, Das et al. (2010, Table 2) numerically identify selected pairs of thresholds for another account with the same optimal portfolio in an example. However, they do not analytically identify *all* pairs of thresholds for the latter account.

[37] Das et al. numerically solve a problem to jointly find the optimal portfolio within a given account and the risk aversion coefficient implied by this portfolio but do not analytically characterize such a coefficient.

are lower and upper bounds on $\gamma^i_{\alpha_m,H_m}$ assuming a threshold return H_m for which the optimal portfolio within account m exists. Let:

$$\overline{\gamma}_{\alpha_m} \equiv \sqrt{\left(z^2_{\alpha_m} - D/C\right) C}. \tag{36}$$

Since z_{α_m} decreases in α_m, $z^2_{\alpha_m} - D/C > 0$, $C > 0$, and $D > 0$, Eq. (36) implies that $\overline{\gamma}_{\alpha_m}$ decreases in α_m. The aforementioned bounds are identified next.

Corollary 7. *Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$. The risk aversion coefficient implied by the optimal portfolio within account m, $\gamma^i_{\alpha_m,H_m}$: (i) converges to zero as H_m converges to minus infinity; and (ii) is $\overline{\gamma}_{\alpha_m}$ if $H_m = \overline{H}_{\alpha_m}$.*

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$. Corollary 7(i) can be seen by recalling that the expected return of the optimal portfolio within account m, E_{α_m,H_m}, converges to infinity as H_m converges to minus infinity; see Corollary 4. Hence, as Fig. 8(a) illustrates, $\gamma^i_{\alpha_m,H_m}$ converges to zero as H_m converges to minus infinity.

Corollary 7(ii) can be seen by recalling that $\boldsymbol{w}_{\alpha_m,H_m}$ equals the global minimum-VaR portfolio at confidence level $1 - \alpha_m$, $\boldsymbol{w}_{1-\alpha_m}$, if $H_m = \overline{H}_{\alpha_m}$; see Corollary 3(ii). Hence, as Fig. 8(a) illustrates, $\gamma^i_{\alpha_m,H_m}$ is $\overline{\gamma}_{\alpha_m}$ if $H_m = \overline{H}_{\alpha_m}$.

As noted earlier, an increase in H_m tightens probability constraint (16) and decreases $\boldsymbol{w}_{\alpha_m,H_m}$'s expected return, E_{α_m,H_m}. Hence, $\gamma^i_{\alpha_m,H_m}$ increases in H_m as Fig. 8(a) illustrates; see Eq. (35). If H_m is notably (slightly) less than $\overline{H}_{\alpha_m}$, then E_{α_m,H_m} notably (slightly) exceeds A/C; see Eqs. (27) and (28). Therefore, $\gamma^i_{\alpha_m,H_m}$ slightly (notably) increases in H_m if H_m is notably (slightly) less than $\overline{H}_{\alpha_m}$ as Fig. 8(a) also illustrates. Intuitively, increasing H_m moves the optimal portfolio within account m *down* along the top part of the mean-variance frontier by a *smaller* extent when H_m is notably less than $\overline{H}_{\alpha_m}$.

Consider an account $m \in \{1, ..., M\}$ with a threshold return $H_m \in (-\infty, A/C)$. Of interest are lower and upper bounds on $\gamma^i_{\alpha_m,H_m}$ assuming a threshold probability α_m for which the optimal portfolio within account m exists. Let:

$$\overline{\gamma}_{H_m} \equiv A - CH_m. \tag{37}$$

Since $C > 0$, Eq. (37) implies that $\overline{\gamma}_{H_m}$ decreases in H_m. The aforementioned bounds are identified next.

Corollary 8. *Consider an account $m \in \{1, ..., M\}$ with threshold return $H_m \in (-\infty, A/C)$. The risk aversion coefficient implied by the optimal portfolio within account m, $\gamma^i_{\alpha_m,H_m}$: (i) converges to zero as the threshold probability α_m converges to $\overline{\alpha}$ from below; and (ii) is $\overline{\gamma}_{H_m}$ if $\alpha_m = \underline{\alpha}_{H_m}$.*

Consider an account $m \in \{1, ..., M\}$ with threshold return $H_m \in (-\infty, A/C)$. Corollary 8(i) can be seen by recalling that the expected return of optimal portfolio within account m, E_{α_m,H_m}, converges to infinity as the threshold probability α_m converges to $\overline{\alpha}$ from below; see Corollary 6. Hence, as Fig. 8(b) illustrates, $\gamma^i_{\alpha_m,H_m}$ converges to zero as α_m converges to $\overline{\alpha}$ from below.

Corollary 8(ii) can be seen by recalling that $\boldsymbol{w}_{\alpha_m,H_m}$ equals the global minimum-VaR portfolio at confidence level $1 - \alpha_m$, $\boldsymbol{w}_{1-\alpha_m}$, if $\alpha_m = \underline{\alpha}_{H_m}$; see Corollary 5(ii). Hence, as Fig. 8(b) illustrates, $\gamma^i_{\alpha_m,H_m} = \overline{\gamma}_{H_m}$ if $\alpha_m = \underline{\alpha}_{H_m}$.

As noted earlier, an increase in α_m loosens probability constraint (16) and increases $\boldsymbol{w}_{\alpha_m,H_m}$'s expected return, E_{α_m,H_m}. Hence, $\gamma^i_{\alpha_m,H_m}$ decreases in α_m as Fig. 8(b) illustrates; see Eq. (35). If α_m is notably (slightly) less than $\overline{\alpha}$, then E_{α_m,H_m} slightly (notably) exceeds A/C; see Eqs. (27) and (28). Therefore, $\gamma^i_{\alpha_m,H_m}$ notably (slightly) decreases in α_m if α_m is notably (slightly) less than $\overline{\alpha}$ as Fig. 8(b) also illustrates. Intuitively, increasing α_m moves the optimal portfolio within account m *up* along the top part of the mean-variance frontier by a *larger* extent when α_m is notably less than $\overline{\alpha}$.

The result that the risk aversion coefficients implied by optimal portfolios within accounts are possibly very sensitive to thresholds has a key practical implication. In order to precisely reflect the levels of risk aversion within the accounts, thresholds should be carefully set, particularly when seeking to reflect relatively high levels of risk aversion.

Given a positive risk aversion coefficient for an investor in Markowitz's model, there are infinitely many pairs of thresholds for an account of a hypothetical investor in Das et al.'s model such that the optimal portfolio within this account of the former investor equals the optimal portfolio of the latter. Some notation is useful to identify these pairs of thresholds. For any risk aversion coefficient $\gamma \in \mathbb{R}_{++}$, let:

$$\overline{\alpha}_\gamma \equiv \Phi\left(-\sqrt{(D+\gamma^2)/C}\right). \tag{38}$$

Since $C > 0$, $D > 0$, and $\gamma > 0$, Eqs. (20) and (38) imply that $\overline{\alpha}_\gamma \in (0, \overline{\alpha})$. For any $\alpha_\gamma \in (0, \overline{\alpha}_\gamma]$, let:

$$H_\gamma \equiv E_\gamma - z_{\alpha_\gamma}\sigma_\gamma. \tag{39}$$

Since z_{α_γ} decreases in α_γ, $z_{\alpha_\gamma} > 0$, and $\sigma_\gamma > 0$, Eq. (39) implies that H_γ increases in α_γ.

Theorem 8. *Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$. The optimal portfolio within an account with threshold probability $\alpha_\gamma \in (0, \overline{\alpha}_\gamma]$ and threshold return of H_γ equals the optimal portfolio of the investor with a risk aversion coefficient of γ.*[38]

Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$. Using Theorem 8, there are infinitely many pairs of thresholds $\{(\alpha_\gamma, H_\gamma)\}_{\alpha \in (0, \overline{\alpha}_\gamma]}$ for an account of a hypothetical investor with the same optimal portfolio as an investor with a risk aversion coefficient of γ. The existence of these pairs of thresholds can be seen by recalling that: (a) the optimal

[38] The condition that $\alpha_\gamma \in (0, \overline{\alpha}_\gamma]$ is similar to Theorem 6's condition that $\alpha \in (0, \overline{\overline{\alpha}}]$; see footnote 35.

portfolio of an investor with a positive risk aversion coefficient is located on the top half of the mean-variance frontier above the global minimum-variance portfolio (Corollary 1(i)); (b) the global minimum-VaR portfolio converges to the global minimum-variance portfolio as the confidence level converges to 100% (see the discussion of Theorem 4); (c) the optimal portfolio within a given account lies on the top half of the mean-variance frontier at the same point as or above the global minimum-VaR portfolio at the confidence level equal to one minus the threshold probability (Corollary 3); and (d) infinitely many pairs of thresholds lead to the same optimal portfolio within a given account (Theorem 6).

3.3. Aggregate portfolio

Suppose that threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$ for any account $m \in \{1, ..., M\}$. Using Theorem 5(i), optimal portfolios within accounts exist and so does the corresponding aggregate portfolio $\boldsymbol{w}_a \equiv \sum_{m=1}^{M} y_m \boldsymbol{w}_{\alpha_m, H_m}$. The next result characterizes the composition, expected return, and standard deviation of this portfolio.

Theorem 9. *If threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$ for any account $m \in \{1, ..., M\}$, then the aggregate portfolio is:*

$$\boldsymbol{w}_a = \theta_a \boldsymbol{w}_0 + (1 - \theta_a) \boldsymbol{w}_1 \tag{40}$$

where $\theta_a \equiv \sum_{m=1}^{M} y_m \theta_{\alpha_m, H_m}$, its expected return is:

$$E_a \equiv \sum_{m=1}^{M} y_m E_{\alpha_m, H_m}, \tag{41}$$

and its standard deviation is:

$$\sigma_a \equiv \sqrt{1/C + (E_a - A/C)^2 / (D/C)}. \tag{42}$$

Suppose that threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$ for any account $m \in \{1, ..., M\}$. Using Eqs. (1) and (40), the aggregate portfolio, $\boldsymbol{w}_a$, is on the mean-variance frontier. The third dot ('•') from the left in Fig. 9 illustrates the location of $\boldsymbol{w}_a$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space when there are three accounts (i.e., $M = 3$). For simplicity, Fig. 9 makes the assumption that $H_3 < H_2 < H_1$ and $\alpha_3 > \alpha_2 > \alpha_1$.[39] Hence, probability constraint (16) loosens when moving from account $m = 1$ to account $m = 2$ and then to account $m = 3$. It follows that $\boldsymbol{w}_{\alpha_3,H_3}$ lies above $\boldsymbol{w}_{\alpha_2,H_2}$, which in turn lies above $\boldsymbol{w}_{\alpha_1,H_1}$ as the first, second, and fourth dots from the left illustrate.

Since $\boldsymbol{w}_a$ is on the mean-variance frontier, $\boldsymbol{w}_a$ solves:

$$\max_{\boldsymbol{w}\in\mathbb{R}^N} \quad E[r_{\boldsymbol{w}}] - (\gamma_a^i/2)\sigma^2[r_{\boldsymbol{w}}] \tag{43}$$

$$s.t. \quad \boldsymbol{w}'\mathbf{1}_N = 1 \tag{44}$$

for some $\gamma_a^i > 0$. Hereafter, γ_a^i is referred to as the *risk aversion coefficient implied by the aggregate portfolio*. The following result characterizes this coefficient.

Theorem 10. *If threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$ for any account $m \in \{1, ..., M\}$, then the risk aversion coefficient implied by the aggregate portfolio is:*

$$\gamma_a^i = \left(\textstyle\sum_{m=1}^M y_m/\gamma_{\alpha_m,H_m}^i\right)^{-1} \tag{45}$$

where γ_{α_m,H_m}^i is given by Eq. (35).

Suppose that threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$ for any account $m \in \{1, ..., M\}$. Using Eq. (45), the risk aversion coefficient implied by the aggregate portfolio, γ_a^i, depends on the fractions of the investor's wealth in the accounts $\{y_m\}_{m=1}^M$ and the thresholds $\{(\alpha_m, H_m)\}_{m=1}^M$ as well as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (through the risk aversion coefficients implied by optimal portfolios within accounts, $\{\gamma_{\alpha_m,H_m}^i\}_{m=1}^M$). Moreover, γ_a^i is a weighted *harmonic* average of $\{\gamma_{\alpha_m,H_m}^i\}_{m=1}^M$ with respective weights of $\{y_m\}_{m=1}^M$.

[39] Graphical results can similarly be obtained if this assumption is not made.

3.4. Differences between the models of Markowitz and Das et al.

Table 1 lists six crucial differences between the models of Markowitz and Das et al. First, an investor in Markowitz's model has one account but an investor in Das et al.'s model has two or more accounts. Second, while the preferences of the former investor are specified by a risk aversion coefficient, the preferences of the latter within a given account are specified by thresholds. Third, the optimal portfolio in Markowitz's model always exists, whereas optimal portfolios within accounts in Das et al.'s model might not exist depending on the thresholds and optimization inputs. Fourth, the optimal portfolio of an investor in the former model (with a positive risk aversion coefficient) lies on the mean-variance frontier above the global minimum-variance portfolio but the optimal portfolio within an account of an investor in the former model (for thresholds such that the portfolio exists) lies on the mean-variance frontier at the same point as or above the global minimum-VaR portfolio at a confidence level equal to one minus the account's threshold probability. Fifth, while different risk aversion coefficients lead to different optimal portfolios in Markowitz's model, infinitely many pairs of thresholds lead to the same optimal portfolio within an account in Das et al.'s model. Sixth, the investor in Markowitz's model has a unique risk aversion coefficient that does not depend on the optimization inputs, whereas the risk aversion coefficients implied by the optimal portfolios within accounts of an investor in Das et al.'s model possibly differ across accounts and depend on the optimization inputs. The risk aversion coefficient implied by the aggregate portfolio of the latter investor (the combination of the investor's optimal portfolios within accounts) also depends on the optimization inputs.

4. Example

This section uses a numerical example to illustrate the theoretical results uncovered in

Sections 2 and 3 for the models of Markowitz and Das et al. It considers the same assets, optimization inputs, and investors as the numerical example of Das et al. By applying the aforementioned theoretical results to their example, numerical results that complement theirs are obtained.

4.1. Assets and optimization inputs

As the first column of Table 2(a) notes, three assets (1, 2, and 3) are available for trade. Asset 1 is analogous to a risky bond, whereas assets 2 and 3 are analogous to, respectively, low- and high-risk stocks. The next two columns indicate that the expected return and standard deviation of asset 3 (high-risk stock) respectively exceed those of asset 2 (low-risk stock), which in turn respectively exceed those of asset 1 (risky bond). The last three columns indicate that the return on asset 1 is uncorrelated with the returns of assets 2 and 3, whereas the returns of assets 2 and 3 are positively correlated.

4.2. Optimal portfolios in Markowitz's model

Consider four investors (1, 2, 3, and 4) in Markowitz's model. As Table 2(b) shows, their risk aversion coefficients are assumed to range from 0.8773 for investor 3 to 3.7950 for investor 1, whereas investor 2 and 4 have risk aversion coefficients of, respectively, 2.7063 and 2.1740. Table 3(a) provides the asset weights, expected returns, and standard deviations of their optimal portfolios. The optimal portfolios of investors with larger risk aversion coefficients have: (i) larger weights in asset 1 (risky bond); (ii) smaller weights in assets 2 and 3 (respectively, low- and high-risk stocks); and (iii) smaller expected returns and standard deviations.

The thin dashed curve in Fig. 10 shows the loss in CER, $L_{\gamma,\gamma_\varepsilon}$, arising from investor 1 using an erroneous risk aversion coefficient of $\gamma_\varepsilon \in (0, 10]$ instead of the 'true' coefficient of

$\gamma = 3.7950$ in seeking to find the optimal portfolio. Note that $L_{\gamma,\gamma_\varepsilon}$ notably decreases in γ_ε if $\gamma_\varepsilon < \gamma$ but increases in γ_ε if $\gamma_\varepsilon > \gamma$. The thick dashed, thin solid, and thick solid curves show similar results for, respectively, investors 2, 3, and 4.[40] Given a sufficiently large value of γ_ε, it can be seen that $L_{\gamma,\gamma_\varepsilon}$ decreases in γ; compare the values of $L_{\gamma,\gamma_\varepsilon}$ for the values of γ of the four investors if, e.g., $\gamma_\varepsilon = 10$. This result can be understood by noting that: (i) given $\gamma > 0$, Eq. (10) implies that $L_{\gamma,\gamma_\varepsilon}$ converges to $(D/C)/(2\gamma)$ as γ_ε converges to infinity as mentioned earlier; and (ii) $(D/C)/(2\gamma)$ decreases in γ.

4.3. Optimal portfolios within accounts and aggregate portfolio in Das et al.'s model

Consider an investor in Das et al.'s model with three accounts (1, 2, and 3). Table 2(c) shows the fractions of the investor's wealth in the accounts (second column) and the thresholds (last two columns).

Fig. 11 examines the existence of optimal portfolios within accounts. The dashed vertical line goes through $\alpha_m = \overline{\alpha} = 33.40\%$; see Eq. (20). The curve plots $\overline{H}_{\alpha_m}$ as a function of $\alpha_m \in (0, \overline{\alpha})$;[41] see Eq. (21). The dots ('•') plot the pairs of thresholds of accounts 1, 2, and 3, $\{(\alpha_m, H_m)\}_{m=1}^{3}$; see the last two columns of Table 2(c). For any account $m \in \{1, 2, 3\}$, note that the pair of thresholds (α_m, H_m) plots *both*: (a) strictly between the y-axis and the dashed vertical line so that $\alpha_m \in (0, \overline{\alpha})$; and (b) below the curve so that $H_m \in (-\infty, \overline{H}_{\alpha_m})$. Hence, optimal portfolios within accounts exist; see Theorem 5(i).

The first three rows of Table 3(b) provide the asset weights, expected returns, standard deviations, and VaRs of the optimal portfolios within accounts as well as the risk aversion coefficients implied by such portfolios.[42] The optimal portfolios within accounts 1, 2, and

[40] The four curves in Fig. 10 are truncated so that $L_{\gamma,\gamma_\varepsilon}$ is 10% or less.
[41] This curve is truncated so that $\overline{H}_{\alpha_m}$ is -16% or more.
[42] While the numerical results in Table 3 reproduce those in Table 1 of Das et al. for completeness, the numerical results in all figures in Section 4 are novel and illustrate the usefulness of the theoretical results uncovered in Sections 2 and 3.

3 equal the optimal portfolios of, respectively, investors 1, 2, and 3 in Markowitz's model; see the second, third, and fourth columns of Tables 3(a) and 3(b). Hence, the risk aversion coefficients implied by these portfolios equal the risk aversion coefficients of such investors; see the last column of Tables 2(b) and 3(b). Since the VaR of the optimal portfolio within a given account is reported at a confidence level equal to one minus the account's threshold probability, the VaR of this portfolio equals minus one multiplied by the account's threshold return; see the last two columns of Table 2(c) and the next to last column of Table 3(b). For example, since account 1 has a threshold probability of 5% and a threshold return of -10%, the optimal portfolio within account 1 has a VaR at the confidence level of 95% $[= 100\% - 5\%]$ of 10% $[= (-1) \times (-10\%)]$.

The last row of Table 3(b) provides the asset weights, expected return, and standard deviation of the aggregate portfolio of the investor in Das et al.'s model (the combination of the investor's optimal portfolios within accounts) as well as the risk aversion coefficient implied by this portfolio. The aggregate portfolio of the investor in Das et al.'s model equals the optimal portfolio of investor 4 in Markowitz's model; see the second, third, and fourth columns of Tables 3(a) and 3(b). Hence, the risk aversion coefficient implied by the aggregate portfolio of the former investor equals the risk aversion coefficient of the latter; see the last column of Tables 2(b) and 3(b).

The thin dashed, thick dashed, and solid curves in Fig. 12 plot the pairs of thresholds $\{(\alpha, H_\alpha)\}_{\alpha \in (0,\overline{\overline{\alpha}})}$ of three accounts with the same optimal portfolios as, respectively, accounts 1, 2, and 3; see Eqs. (31) and (32) as well as Theorem 6.[43] The dots ('•') represent the pairs of thresholds of accounts 1, 2, and 3; see the last two columns of Table 2(c). Two results can be seen. First, the thin dashed curve lies above the thick dashed curve, which in turn

[43] The solid curve is truncated so that H_α is -100% or more.

lies above the solid curve. This result can be understood by noting that: (a) given α_m, the optimal portfolio within account m moves down the top half of the mean-variance frontier as H_m increases; and (b) the optimal portfolio within account 1 lies on the top half of the mean-variance frontier below the optimal portfolio within account 2, which in turn lies below the optimal portfolio within account 3.

Second, the curves are close to each other for relatively large thresholds. This result can be understood by noting that H_α is less sensitive to the account m used in the right-hand side of Eq. (32) if α is relatively large. For example, the values of H_α for accounts 1, 2, and 3 are, respectively: (i) -5.53%, -9.06%, and -36.61% if $\alpha = 10\%$; and (ii) 3.78%, 3.49%, and 0.59% if $\alpha = 30\%$. The fact that the curves are close to each other for relatively large thresholds implies that three accounts with certain slightly different pairs of thresholds would have the same optimal portfolios as, respectively, accounts 1, 2, and 3. For example, three accounts with threshold probabilities of 30%, 31%, and 32% as well as threshold returns of 3.78%, 3.95%, and 3.37% would have the same optimal portfolios as, respectively, accounts 1, 2, and 3. Since the optimal portfolios within the latter accounts notably differ as Table 3(b) shows, so do the optimal portfolios within the former. A key practical implication follows. In attempting to infer the preferences of an investor with thresholds to implement Das et al.'s model, practitioners should be aware that the use of relatively large thresholds leads to optimal portfolios within accounts that are *very* sensitive to the thresholds. Hence, thresholds should be carefully chosen so that these portfolios are properly identified.

Given a threshold probability of $\alpha_1 = 5\%$, Fig. 13(a) plots the risk aversion coefficient implied by the optimal portfolio within account 1, $\gamma^i_{\alpha_1,H_1}$, as a function of threshold return H_1. Note that $\gamma^i_{\alpha_1,H_1}$: (i) is strictly between zero and $\overline{\gamma}_{\alpha_1} = 32.78$; and (ii) slightly (notably) increases in H_1 for relatively small (large) values of H_1. Given a threshold return of $H_1 = -10\%$, Fig. 13(b) plots $\gamma^i_{\alpha_1,H_1}$ as a function of threshold probability of α_1. Note that $\gamma^i_{\alpha_1,H_1}$: (i) is strictly between zero and $\overline{\gamma}_{H_1} = 65.52$; and (ii) notably (slightly) decreases in α_1 for relatively small (large) values of α_1. Similar results hold for account 2 in Figs. 13(c) and 13(d) as well as for account 3 in Figs. 13(e) and 13(f).[44] In sum, the risk aversion coefficients implied by optimal portfolios within accounts are very sensitive to the thresholds if such coefficients are relatively large but notably less so if the coefficients are relatively small.

5. Conclusion

Following the insights of Markowitz (1952), modern portfolio theory considers investors whose preferences are represented by a function defined over the means and variances of portfolio returns and specified with a risk aversion coefficient. In Das et al. (2010), however, Markowitz and three co-authors note that modern portfolio theory does not address the practical fact that an investor might have: (i) various investing motives such as retirement and bequest; (ii) preferences that are difficult to precisely specify with a risk aversion coefficient; and (iii) different goals for different investing motives. Accordingly, their model considers an investor whose wealth is divided among mental accounts (hereafter, 'accounts') with different investing motives. The investor's preferences within a given account are specified by a threshold probability and a threshold return (hereafter, 'thresholds'). Formally, the optimal portfolio within the account maximizes the account's expected return subject to the constraint that the probability of the account's return being less than or equal to the threshold return does not exceed the threshold probability. Hence, the account's Value-at-Risk (VaR) at the confidence level of one minus the threshold probability cannot exceed

[44] Since $\alpha_1 < \alpha_2 < \alpha_3$, Figs. 13(a), 13(c), and 13(e) show that $\overline{\gamma}_{\alpha_1} > \overline{\gamma}_{\alpha_2} > \overline{\gamma}_{\alpha_3}$; recall that $\overline{\gamma}_{\alpha_m}$ decreases in α_m (see Eq. (36)). Similarly, since $H_3 < H_1 < H_2$, Figs. 13(b), 13(d), and 13(f) show that $\overline{\gamma}_{H_3} > \overline{\gamma}_{H_1} > \overline{\gamma}_{H_2}$; recall that $\overline{\gamma}_{H_m}$ decreases in H_m (see Eq. (37)).

minus one multiplied by the threshold return. Reflecting the fact that different accounts have different investment motives, thresholds possibly vary across accounts.

The models of Markowitz and Das et al. are closely related if asset returns are assumed to have a multivariate Normal distribution and thresholds are such that optimal portfolios within accounts exist in Das et al.'s model. Since the VaR of a portfolio is a linear function of its mean and standard deviation under this distribution, optimal portfolios within accounts are on the *mean-variance frontier*. A portfolio is on the mean-variance frontier if there is no portfolio with the same mean and a smaller variance. The optimal portfolio within a given account of an investor in Das et al.'s model would be selected by a hypothetical investor in Markowitz's model with some (implied) risk aversion coefficient. Conversely, the optimal portfolio of an investor in the latter model would be selected within an account of a hypothetical investor in the former with some (implied) thresholds. The aggregate portfolio of an investor in Das et al.'s model (the combination of the investor's optimal portfolios within accounts) is also on the mean-variance frontier if short selling is allowed.

Bridging Markowitz's contributions, results along three dimensions are here uncovered. First, consider the loss in certainty-equivalent return (CER) arising from an investor using an erroneous (instead of the 'true') risk aversion coefficient in seeking to find the optimal portfolio in Markowitz's model. An analytical expression for the loss in CER is derived. Importantly, the loss in CER in the case where the erroneous coefficient is less than the 'true' coefficient by some amount *exceeds* the loss in CER in the case where the erroneous coefficient is greater than the 'true' coefficient by the same amount. Moreover, the loss in CER is unbounded (bounded) from above in the former (latter) case.

Second, consider the impact of using thresholds instead of a risk aversion coefficient to specify investor preferences on portfolio selection. For any positive risk aversion coefficient, the optimal portfolio in Markowitz's model lies on the mean-variance frontier above the global minimum-variance portfolio. In comparison, for any pair of thresholds such that the optimal portfolio within a given account exists in Das et al.'s model, this portfolio lies on the mean-variance frontier at the same point as or above the global minimum-VaR portfolio at the confidence level of one minus the threshold probability, which in turn lies above the global minimum-variance portfolio. While different risk aversion coefficients lead to different optimal portfolios in Markowitz's model, infinitely many pairs of thresholds lead to the same optimal portfolio within an account in Das et al.'s model.

Third, consider the risk aversion coefficients implied by optimal portfolios within accounts when these portfolios exist. Using an analytical characterization of such coefficients, key observations are made. Assuming thresholds such that the optimal portfolio within a given account exists, the risk aversion coefficient implied by this portfolio: (i) is strictly between zero and a positive value associated with the global minimum-VaR portfolio at the confidence level of one minus the threshold probability; (ii) decreases in the threshold probability; (iii) increases in the threshold return; and (iv) is very sensitive to the thresholds if the coefficient is relatively large but notably less so if the coefficient is relatively small.

The results uncovered here have two key practical implications. First, in attempting to infer the preferences of an investor with a risk aversion coefficient to implement Markowitz's model, practitioners should be aware that the loss in CER arising from using an erroneous coefficient less than the 'true' coefficient by some amount *exceeds* the loss in CER arising from using an erroneous coefficient greater than the 'true' coefficient by the same amount. Second, in attempting to infer the preferences of an investor with thresholds to implement

Das et al.'s model, practitioners should be aware that optimal portfolios within accounts and the risk aversion coefficients implied by such portfolios are *very* sensitive to the thresholds for relatively large coefficients but notably less so for relatively small coefficients.

Disclosure of interest

There are no relevant financial or non-financial competing interests to report.

Data availability statement

Data sharing not applicable – no new data generated.

Declaration of funding

No funding was received.

References

Alexander, G. J. 2009. "From Markowitz to Modern Risk Management." *The European Journal of Finance* 15: 451–461. doi: 10.1080/13518470902853566.

Alexander, G. J., and A. M. Baptista. 2002. "Economic Implications of Using a Mean-VaR Model for Portfolio Selection: A Comparison with Mean-Variance Analysis." *Journal of Economic Dynamics and Control* 26: 1159–1193. doi: 10.1016/S0165-1889(01)00041-0.

Alexander, G. J., A. M. Baptista, and S. Yan. 2020. "Portfolio Selection with Mental Accounts: An Equilibrium Model with Endogenous Risk Aversion." *Journal of Banking and Finance* 110: 105599. doi: 10.1016/j.jbankfin.2019.07.019.

Arzac, E. R., and V. S. Bawa. 1977. "Portfolio Choice and Equilibrium in Capital Markets with Safety-First Investors." *Journal of Financial Economics* 4: 277–288. doi: 10.1016/0304-405X(77)90003-4.

Baumol, W. J. 1963. "An Expected Gain-Confidence Limit Criterion for Portfolio Selection." *Management Science* 10: 174-182. doi: 10.1287/mnsc.10.1.174.

Black, F. 1972. "Capital Market Equilibrium with Restricted Borrowing." *The Journal of Business* 45: 444–455. doi: www.jstor.org/stable/2351499.

Das, S., H. M. Markowitz, J. Scheid, and M. Statman. 2010. "Portfolio Optimization with Mental Accounts." *Journal of Financial and Quantitative Analysis* 45: 311–334. doi: 10.1017/S0022109010000141.

Das, S., H. M. Markowitz, J. Scheid, and M. Statman. 2011. "Portfolios for Investors Who Want to Reach Their Goals While Staying on the Mean–Variance Efficient Frontier." *The Journal of Wealth Management* 14: 25–31. doi: 10.3905/jwm.2011.14.2.025.

Das, S., and M. Statman. 2013. "Options and Structured Products in Behavioral Portfolios." *Journal of Economic Dynamics and Control* 37: 137–153. doi: 10.1016/j.jedc.2012.07.004.

Elton, E. J., M. J. Gruber, S. J. Brown, and W. N. Goetzmann. 2014. *Modern Portfolio Theory and Investment Analysis.* Hoboken, NJ: John Wiley & Sons.

Fabozzi, F. J., D. Huang, and G. Zhou. 2010. "Robust Portfolios: Contributions from Operations Research and Finance." *Annals of Operations Research* 176: 191–220. doi: 10.1007/s10479-009-0515-6.

Goetzmann, W. N. 2023. "Harry Markowitz in Memoriam." *Financial Analysts Journal*, 79: 5–7, doi: 10.1080/0015198X.2023.2251861.

Huang, C., and R. H. Litzenberger. 1988. *Foundations for Financial Economics*. Englewood Cliffs, NJ: Prentice-Hall.

Hull, J. 2023. *Risk Management and Financial Institutions.* Hoboken, NJ: John Wiley & Sons.

Ingersoll, J. E. 1987. *Theory of Financial Decision Making*. Savage, MD: Rowman & Littlefield Publishers.

Koumou, G. B. 2020. "Diversification and Portfolio Theory: A Review." *Financial Markets* *and Portfolio Management* 34: 267–312. doi: 10.1007/s11408-020-00352-6.

Levy, H., and Markowitz, H. M. 1979. "Approximating Expected Utility by a Function of Mean and Variance." *The American Economic Review* 69: 308–31.
doi: www.jstor.org/stable/1807366.

Markowitz, H. M. 1952. "Portfolio Selection." *Journal of Finance* 7: 77–91.
doi: 10.2307/2975974.

Markowitz, H. M. 1959. *Portfolio Selection: Efficient Diversification of Investments*. New Haven, CT: Yale University Press.

Markowitz, H. M. 1999. "The Early History of Portfolio Theory: 1600-1960." *Financial Analysts Journal* 55: 5–16. doi: 10.2469/faj.v55.n4.2281.

Markowitz, H. M. 2000. *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. New York: John Wiley & Sons.

Markowitz, H. M. 2009. *Selected Works*. Hackensack, NJ: World Scientific Publishing Company.

Markowitz, H. M. 2010. "Portfolio Theory: As I Still See It." *Annual Review of Financial Economics* 2: 1–23. doi: 10.1146/annurev-financial-011110-134602.

O'cinneide, C. A. 1990. "The Mean Is within One Standard Deviation of Any Median." *The American Statistician* 44, 292–294. doi: 10.1080/00031305.1990.10475743.

Roll, R. 1992. "A Mean/Variance Analysis of Tracking Error." *The Journal of Portfolio Management* 18: 13–22. doi: 10.3905/jpm.1992.701922.

Roy, A. D. 1952. "Safety First and the Holding of Assets." *Econometrica* 20: 431–449.
doi: 10.2307/1907413.

Rubinstein, M. 2002. "Markowitz's 'Portfolio Selection': A Fifty-Year Retrospective." *Journal of Finance* 57: 1041–45. doi: 10.1111/1540-6261.00453.

Sharpe, W. F. 1964. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance* 19: 425–442. doi: 10.1111/j.1540-6261.1964.tb02865.x.

Shefrin, H., and M. Statman. 2000. "Behavioral Portfolio Theory." *Journal of Financial and Quantitative Analysis* 35: 127–151. doi: 10.2307/2676187.

Statman, M. 2017. *Finance for Normal People: How Investors and Markets Behave.* Oxford University Press, New York.

Statman, M. 2024. "Harry Markowitz's Two Intellectual Children: Mean–Variance and Behavioral Portfolio Theories." *The Journal of Portfolio Management* 50: 24–29.
doi: 10.3905/jpm.2024.50.8.024.

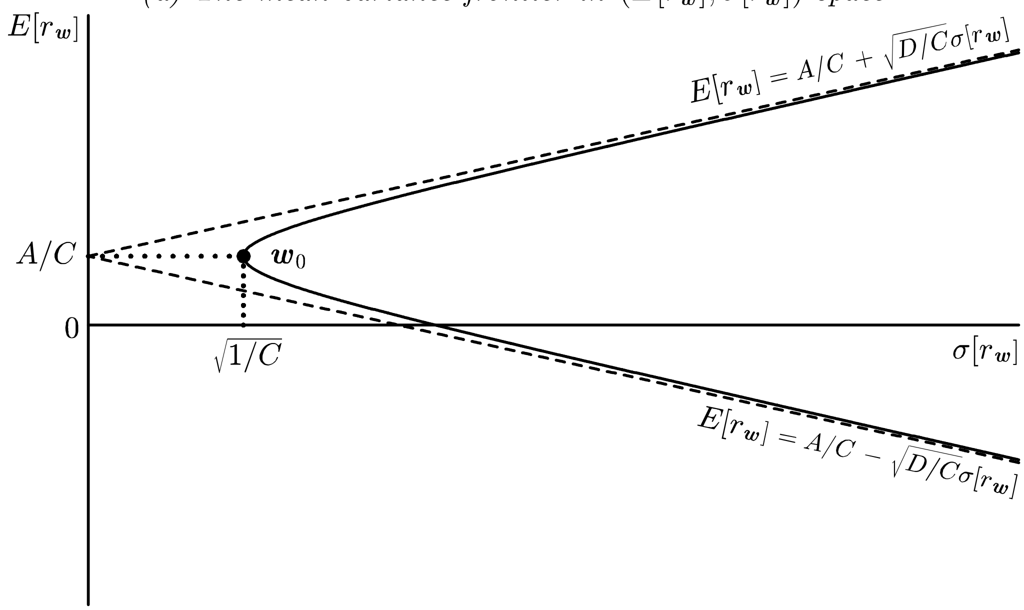
Telser, L. G. 1955. "Safety First and Hedging." *The Review of Economic Studies* 23: 1–16.
doi: 10.2307/2296146.

Tobin, J. 1958. "Liquidity Preference as Behavior Towards Risk." *The Review of Economic Studies*, 25: 65–86, doi: 10.2307/2296205.

Fig. 1. The mean-variance frontier

In Fig. 1(a), the hyperbola plots portfolios on the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. The dashed half-lines show the asymptotes of the frontier: $E[r_{\boldsymbol{w}}] = A/C \pm \sqrt{D/C}\sigma[r_{\boldsymbol{w}}]$. The dot ('•') represents the global minimum-variance portfolio, $\boldsymbol{w}_0$. Its expected return and standard deviation are, respectively, A/C and $\sqrt{1/C}$. In Fig. 1(b), the parabola plots portfolios on the frontier in $(E[r_{\boldsymbol{w}}], \sigma^2[r_{\boldsymbol{w}}])$ space. The leftmost and rightmost dots represent, respectively, portfolios $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$. While their respective expected returns are A/C and B/A, their respective variances are $1/C$ and B/A^2. Note that $\boldsymbol{w}_1$ lies in $(E[r_{\boldsymbol{w}}], \sigma^2[r_{\boldsymbol{w}}])$ space at the point where a ray from the origin that goes through $\boldsymbol{w}_0$ crosses the frontier; see Section 2 for the definition of A, B, C, and D.

(a) The mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space



(b) The mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma^2[r_{\boldsymbol{w}}])$ space

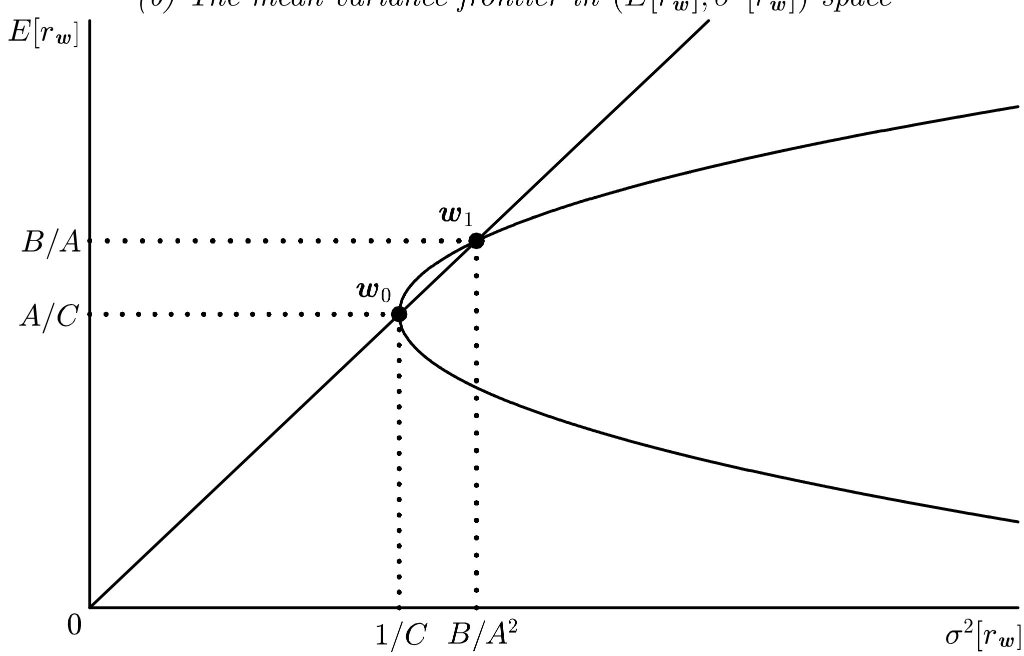
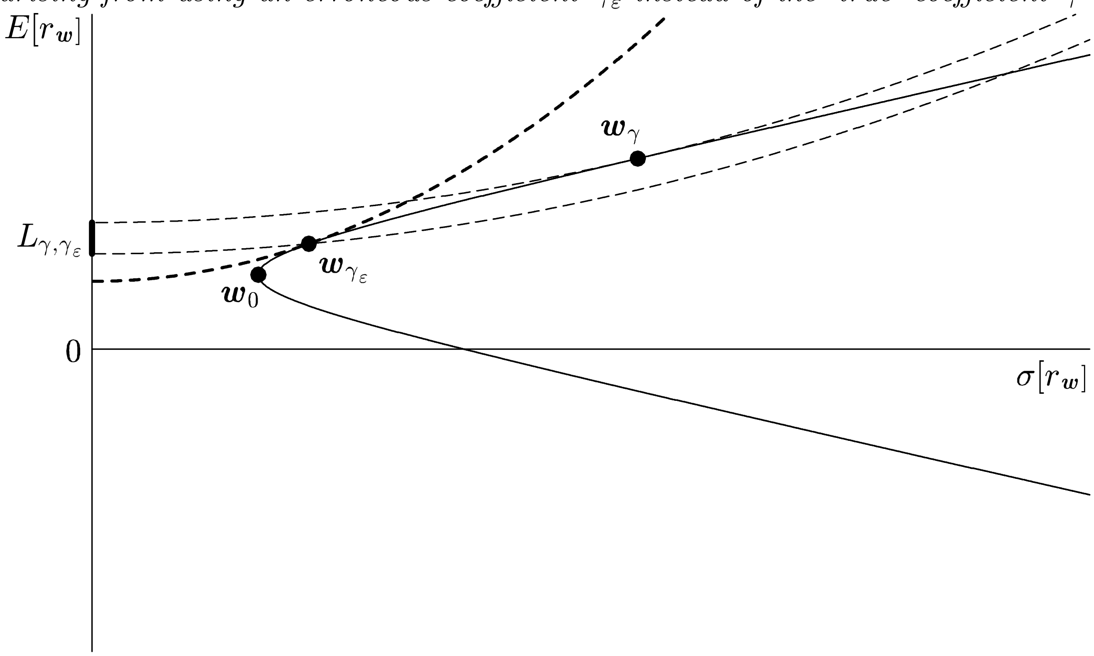


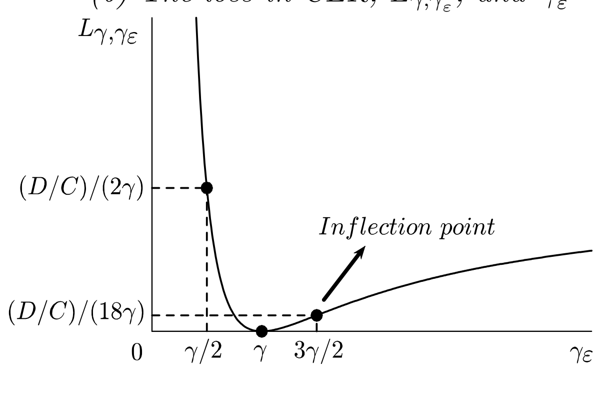
Fig. 2. The optimal portfolio and the loss in CER arising from using an erroneous risk aversion coefficient instead of the 'true' risk aversion coefficient

Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$. In Fig. 2(a), the rightmost dot ('•') shows that the investor's optimal portfolio, $\boldsymbol{w}_\gamma$, lies in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space at the point where the top thin dashed indifference curve associated with γ is tangent to the top half of the curve representing portfolios on the mean-variance frontier. The leftmost dot plots the global minimum-variance portfolio, $\boldsymbol{w}_0$. The middle dot shows that the portfolio selected by the investor when using an erroneous risk aversion coefficient of $\gamma_\varepsilon > \gamma$, $\boldsymbol{w}_{\gamma_\varepsilon}$, lies in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space at the point where the thick dashed indifference curve associated with γ_ε is tangent to the top half of the mean-variance frontier. The bottom thin dashed indifference curve associated to γ goes through $\boldsymbol{w}_{\gamma_\varepsilon}$. The loss in CER arising from the investor selecting $\boldsymbol{w}_{\gamma_\varepsilon}$ instead of $\boldsymbol{w}_\gamma$, $L_{\gamma,\gamma_\varepsilon}$, is the vertical distance between the top and bottom thin dashed curves. Given $\gamma \in \mathbb{R}_{++}$, Fig. 2(b) illustrates how $L_{\gamma,\gamma_\varepsilon}$ depends on γ_ε. Fig. 2(c) reports the relative change in the loss in CER if γ_ε is less than γ by some percentage $\kappa \in (0, 30\%)$ instead of being greater than γ by the same percentage, RCL_κ.

(a) The optimal portfolio with a risk aversion coefficient of γ and the loss in CER arising from using an erroneous coefficient γ_ε instead of the 'true' coefficient γ



(b) The loss in CER, $L_{\gamma,\gamma_\varepsilon}$, and γ_ε



(c) The relative change in the loss in CER, RCL_κ, if γ_ε is less (instead of being greater) than γ by percentage κ

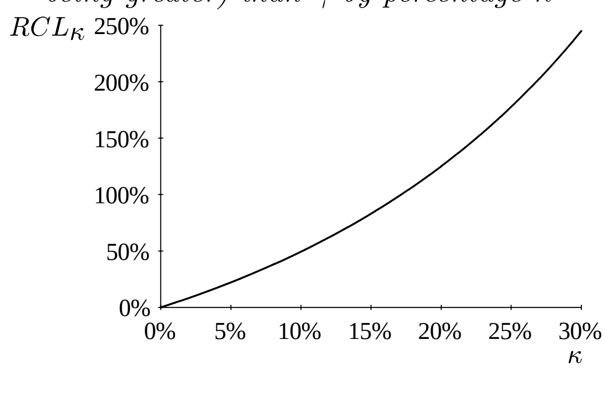
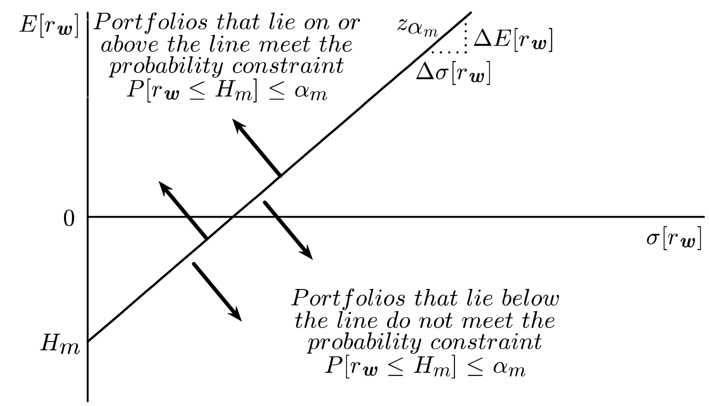


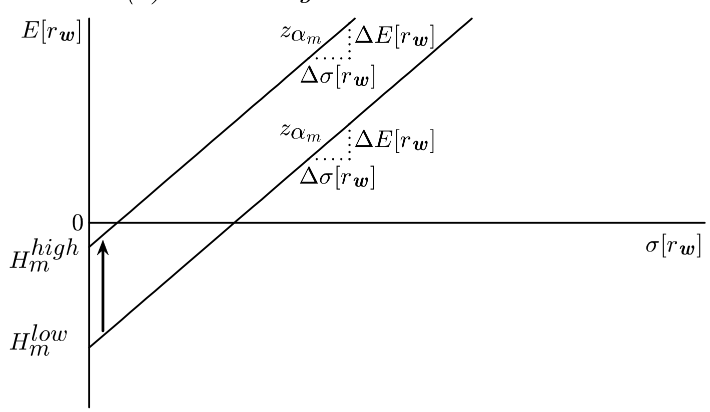
Fig. 3. Portfolios that meet probability constraint (16) and the size of the thresholds

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, 1/2)$ and threshold return $H_m \in \mathbb{R}$. Let $z_{\alpha_m} = -\Phi^{-1}(\alpha_m)$ where $\Phi(\cdot)$ is the cumulative standard Normal univariate distribution function. In Fig. 3(a), the line has an intercept of H_m and a slope of z_{α_m} $[= \triangle E[r_{\boldsymbol{w}}]/\triangle\sigma[r_{\boldsymbol{w}}]]$. Portfolios that lie on or above this line meet probability constraint (16). Portfolios that lie below the line do not meet this constraint. Given a threshold probability of α_m, Fig. 3(b) shows that increasing the threshold return from H_m^{low} to H_m^{high} tightens the constraint because the intercept of the line associated with the constraint increases from H_m^{low} to H_m^{high}. Given a threshold return of H_m, Fig 3(c) shows that increasing the threshold probability from α_m^{low} to α_m^{high} loosens the constraint because the slope of the line associated with the constraint decreases from $z_{\alpha_m^{low}}$ to $z_{\alpha_m^{high}}$ (recall that z_α decreases in α).

(a) Portfolios that meet probability constraint (16)



(b) Increasing the threshold return



(c) Increasing the threshold probability

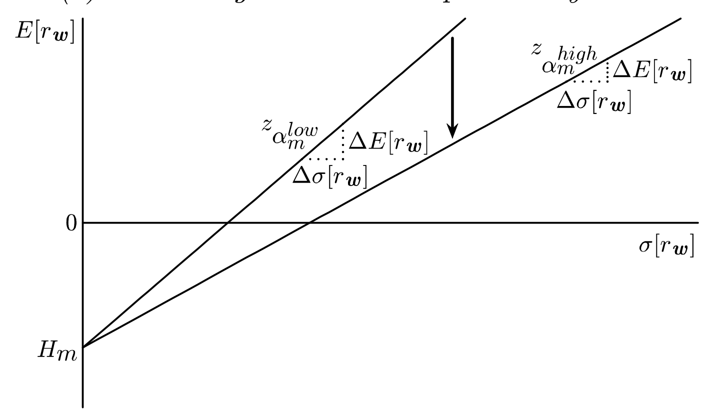


Fig. 4. The global minimum-VaR portfolio

The curve represents portfolios on the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. The leftmost dot ('•') plots the global minimum-variance portfolio, $\boldsymbol{w}_0$. Its expected return and standard deviation are, respectively, A/C and $\sqrt{1/C}$. The rightmost dot plots the global minimum-VaR portfolio, $\boldsymbol{w}_{1-\alpha}$, when the confidence level is $1-\alpha$ and $\alpha \in (0, \overline{\alpha})$; see Eq. (20). Its expected return, standard deviation, and VaR at confidence level $1-\alpha$ are, respectively, $E_{1-\alpha}$, $\sigma_{1-\alpha}$, and $V_{1-\alpha}$; see Eqs. (23), (24), and (25). Portfolio $\boldsymbol{w}_{1-\alpha}$ lies at the point where a line with slope z_α $[= \triangle E[r_{\boldsymbol{w}}]/\triangle\sigma[r_{\boldsymbol{w}}]]$ is tangent to the top half of the mean-variance frontier. The intercept of this line is $\overline{H}_\alpha = -V_{1-\alpha}$; see Eqs. (21) and (25).

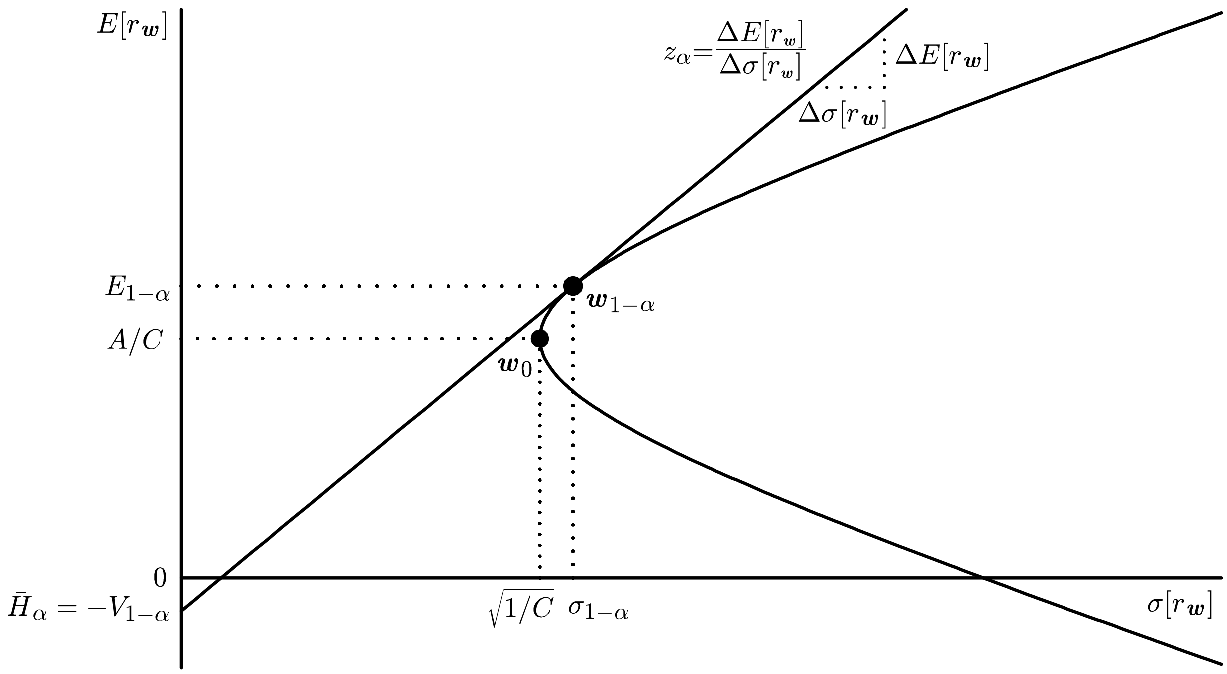
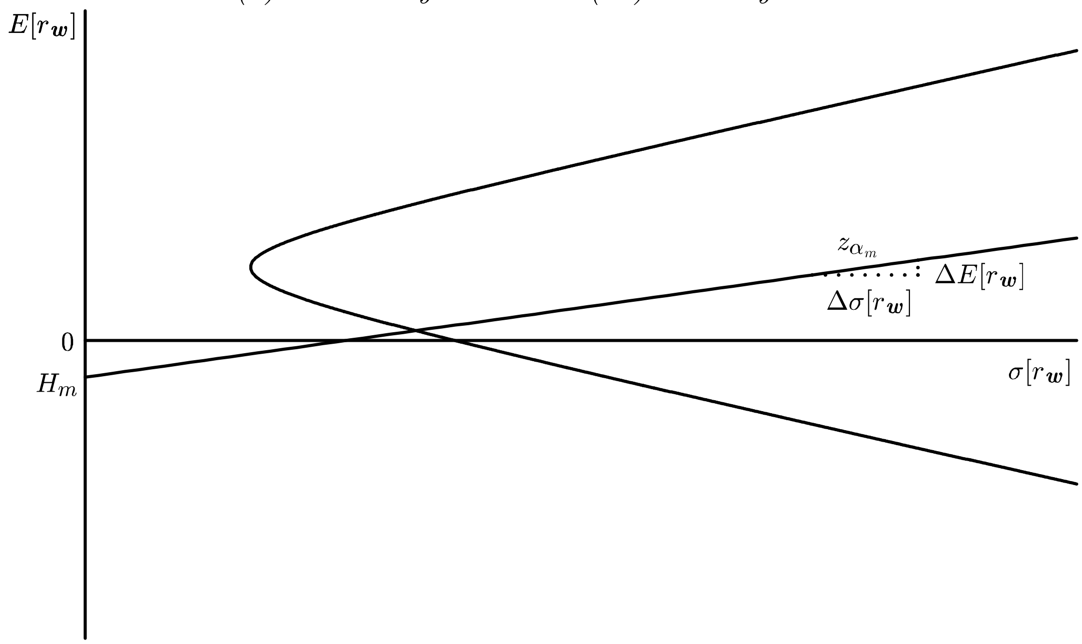


Fig. 5. Non-existence of the optimal portfolio within a given account

The curve represents portfolios on the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, 1/2)$ and threshold return $H_m \in \mathbb{R}$. The line has an intercept of H_m and a slope of z_{α_m} $[= \triangle E[r_{\boldsymbol{w}}]/\triangle\sigma[r_{\boldsymbol{w}}]]$. Portfolios that meet probability constraint (16) lie on or above this line. In Fig. 5(a), the constraint is overly *loose*. Since the portfolios on the frontier with sufficiently large expected returns lie above the solid line, the set of expected returns of portfolios that meet the constraint does not have a finite upper bound and thus the optimal portfolio within account m does not exist. In Fig. 5(b), the constraint is overly *tight*. Since the solid line lies above the curve, no portfolio meets the constraint and thus the optimal portfolio within account m does not exist.

(a) Probability constraint (16) is overly loose



(b) Probability constraint (16) is overly tight

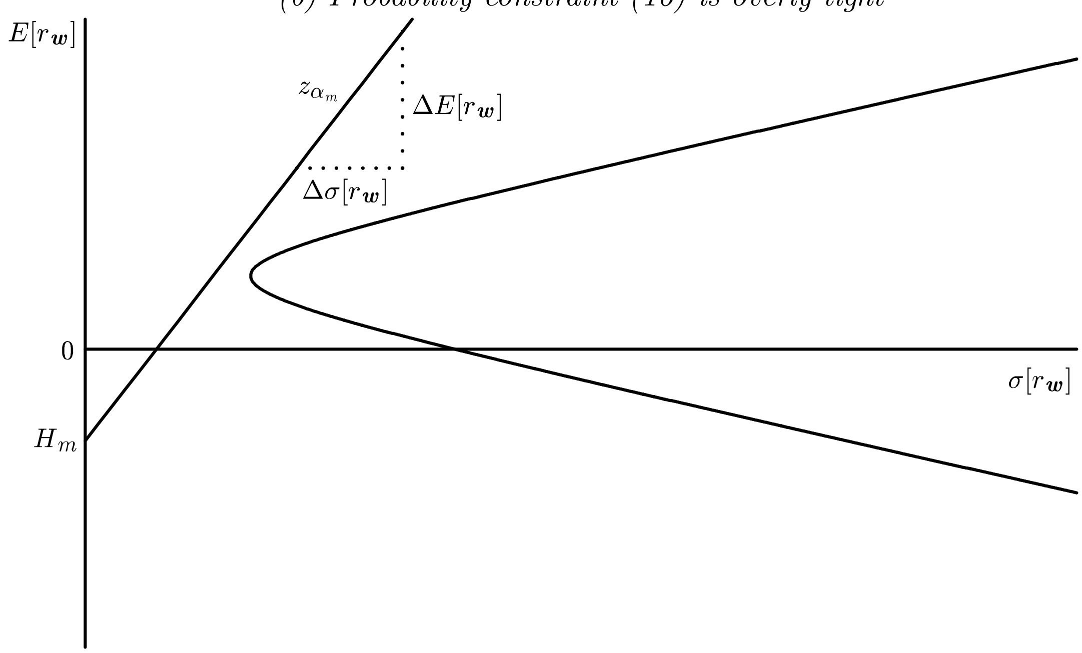
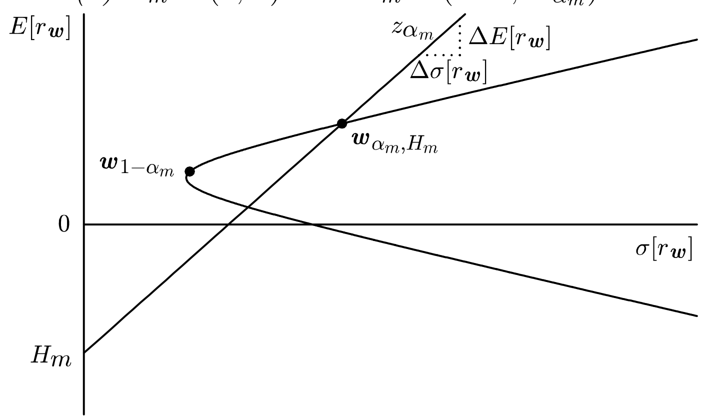


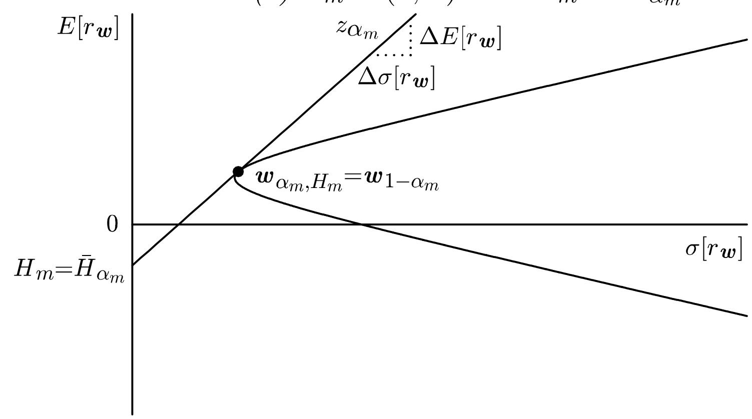
Fig. 6. The optimal portfolio within account m

Consider an account $m \in \{1, ..., M\}$ with threshold probability α_m, threshold return H_m, and optimal portfolio $\boldsymbol{w}_{\alpha_m,H_m}$. In Figs. 6(a) through 6(d), the curve represents portfolios on the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space, whereas the line has an intercept of H_m and a slope of z_{α_m} $[= \triangle E[r_{\boldsymbol{w}}]/\triangle\sigma[r_{\boldsymbol{w}}]]$. In Fig. 6(a), $\alpha_m \in (0, \overline{\alpha})$, $H_m \in (-\infty, \overline{H}_{\alpha_m})$, the rightmost dot ('•') represents $\boldsymbol{w}_{\alpha_m,H_m}$, and the leftmost dot represents the global minimum-VaR portfolio at confidence level $1 - \alpha_m$, $\boldsymbol{w}_{1-\alpha_m}$. In Fig. 6(b), $\alpha_m \in (0, \overline{\alpha})$, $H_m = \overline{H}_{\alpha_m}$, and the dot represents $\boldsymbol{w}_{\alpha_m,H_m} = \boldsymbol{w}_{1-\alpha_m}$. In Fig 6(c), $\alpha_m \in (\underline{\alpha}_{H_m}, \overline{\alpha})$, $H_m \in (-\infty, A/C)$, the rightmost dot represents $\boldsymbol{w}_{\alpha_m,H_m}$, and the leftmost dot represents the global minimum-VaR portfolio at confidence level $1 - \underline{\alpha}_{H_m}$, $\boldsymbol{w}_{1-\underline{\alpha}_{H_m}}$. In Fig. 6(d), $\alpha_m = \underline{\alpha}_{H_m}$, $H_m \in (-\infty, A/C)$, and the dot represents $\boldsymbol{w}_{\alpha_m,H_m} = \boldsymbol{w}_{1-\underline{\alpha}_{H_m}}$.

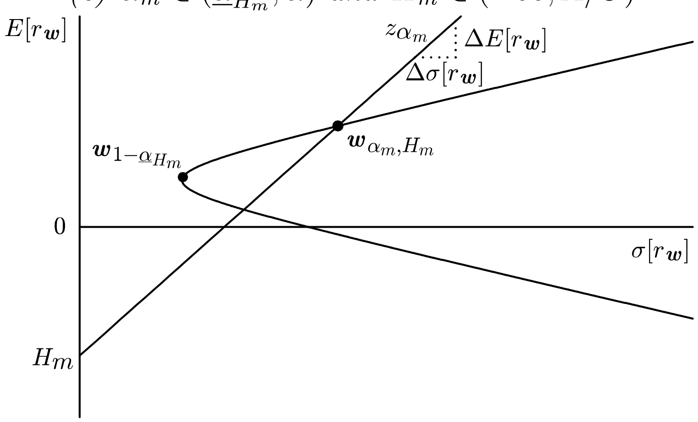
(a) $\alpha_m \in (0, \overline{\alpha})$ *and* $H_m \in (-\infty, \overline{H}_{\alpha_m})$



(b) $\alpha_m \in (0, \overline{\alpha})$ *and* $H_m = \overline{H}_{\alpha_m}$



(c) $\alpha_m \in (\underline{\alpha}_{H_m}, \overline{\alpha})$ *and* $H_m \in (-\infty, A/C)$



(d) $\alpha_m = \underline{\alpha}_{H_m}$ *and* $H_m \in (-\infty, A/C)$

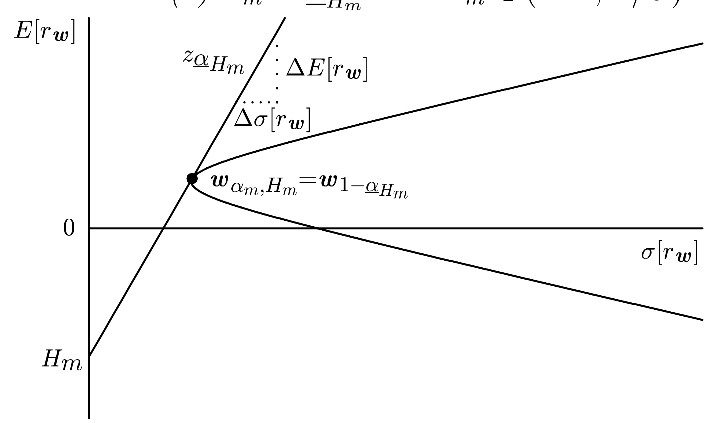
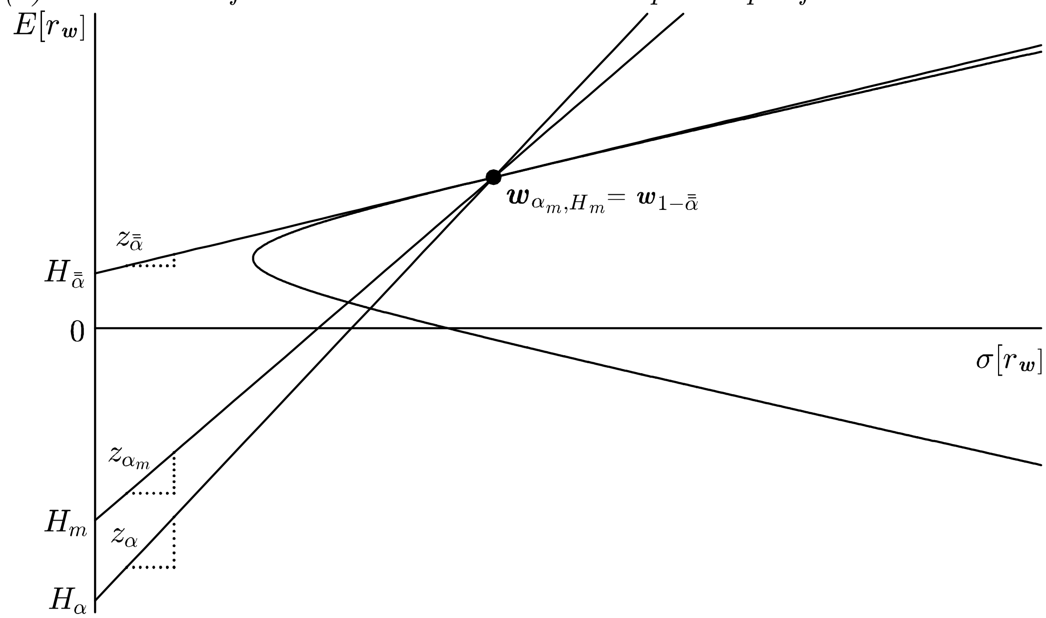


Fig. 7. Thresholds of accounts with the same optimal portfolio as account m

Consider an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$, threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, and optimal portfolio $\boldsymbol{w}_{\alpha_m,H_m}$. As the dot ('•') in Fig. 7(a) shows, $\boldsymbol{w}_{\alpha_m,H_m}$ lies in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space at the point where a line with an intercept of H_m and a slope of z_{α_m} crosses the top half of the curve representing portfolios on the mean-variance frontier. The optimal portfolio within an account with thresholds $\alpha \in (0, \overline{\alpha})$ and H_α is $\boldsymbol{w}_{\alpha_m,H_m}$ because it lies at the point where a line with an intercept of H_α and a slope of z_α crosses the top half of the frontier. The optimal portfolio within an account with thresholds $\overline{\overline{\alpha}}$ and $H_{\overline{\overline{\alpha}}}$ is again $\boldsymbol{w}_{\alpha_m,H_m}$ because it lies at the point where a line with an intercept of $H_{\overline{\overline{\alpha}}}$ and a slope of $z_{\overline{\overline{\alpha}}}$ is tangent to the top half of the frontier. Hence, $\boldsymbol{w}_{\alpha_m,H_m}$ equals the global minimum-VaR portfolio at confidence level $1 - \overline{\overline{\alpha}}$, $\boldsymbol{w}_{1-\overline{\overline{\alpha}}}$. Fig. 7(b) plots the thresholds $\{(\alpha, H_\alpha)\}_{\alpha \in (0, \overline{\overline{\alpha}}]}$ of all accounts with the same optimal portfolio as account m; see Eqs. (31) and (32).

(a) Thresholds of two accounts with the same optimal portfolio as account m



(b) Thresholds of all accounts with the same optimal portfolio as account m

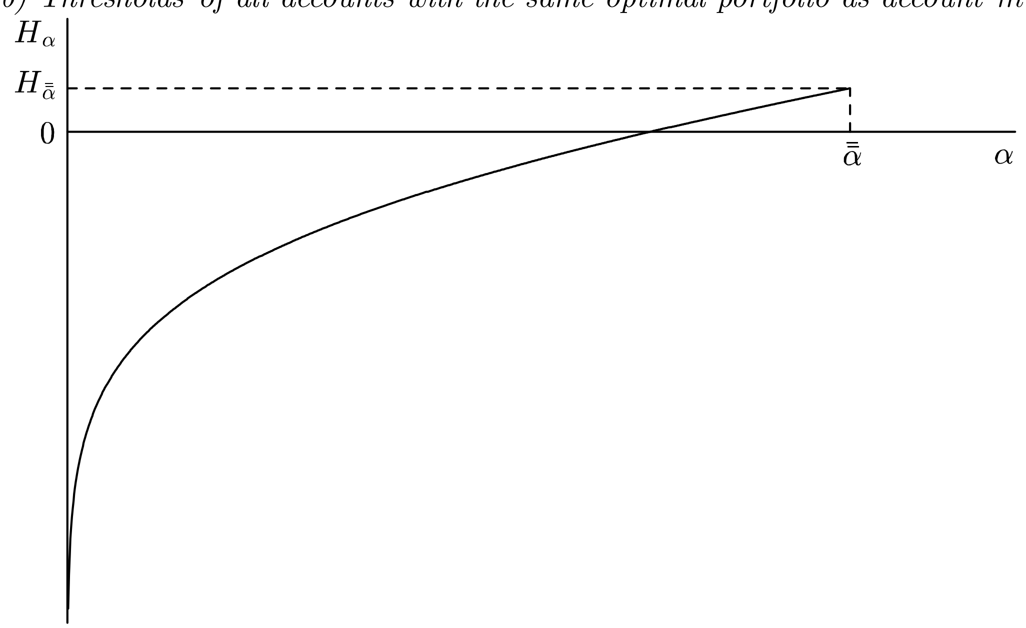
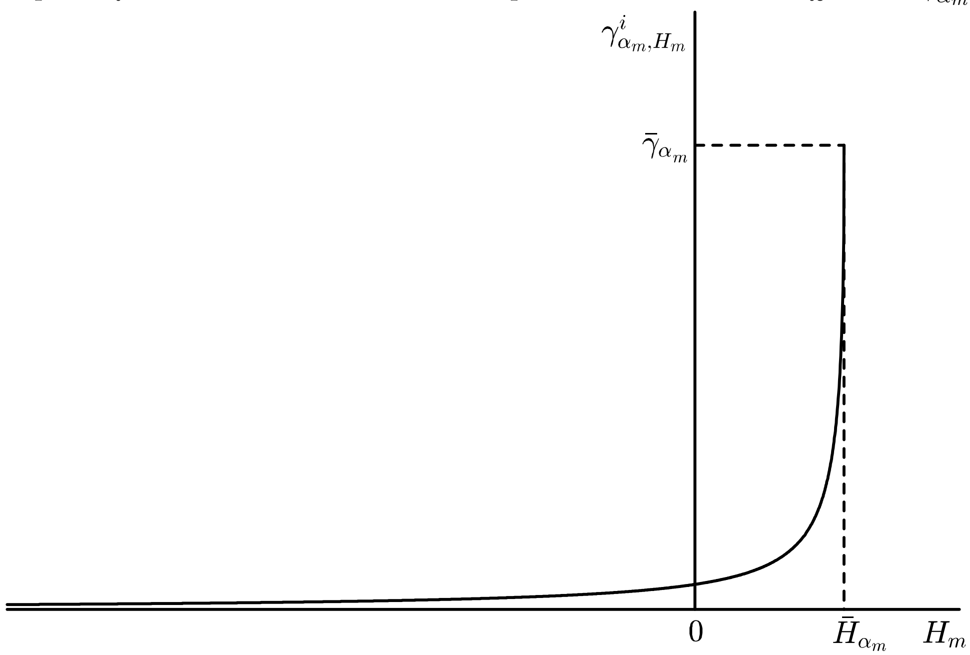


Fig. 8. Impact of the thresholds on the risk aversion coefficient implied by the optimal portfolio within account m

Consider an account $m \in \{1, ..., M\}$. Given a threshold probability $\alpha_m \in (0, \overline{\alpha})$, Fig. 8(a) plots the risk aversion coefficient implied by the optimal portfolio within account m, $\gamma^i_{\alpha_m,H_m}$, as a function of threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$; see Eqs. (20) and (21). Note that $\gamma^i_{\alpha_m,H_m}$: (i) converges to zero as H_m converges to minus infinity; and (ii) is $\overline{\gamma}_{\alpha_m}$ if $H_m = \overline{H}_{\alpha_m}$; see Eq. (36) and Corollary 7. Given a threshold return $H_m \in (-\infty, A/C)$, Fig. 8(b) plots $\gamma^i_{H_m,\alpha_m}$ as a function of threshold probability $\alpha_m \in [\underline{\alpha}_{H_m}, \overline{\alpha})$; see Eq. (30). Note that $\gamma^i_{\alpha_m,H_m}$: (i) converges to zero as α_m converges to $\overline{\alpha}$ from below; and (ii) is $\overline{\gamma}_{H_m}$ if $\alpha_m = \underline{\alpha}_{H_m}$; see Eq. (37) and Corollary 8.

(a) Impact of threshold return H_m on implied risk aversion coefficient $\gamma^i_{\alpha_m,H_m}$



(b) Impact of threshold probability α_m on implied risk aversion coefficient $\gamma^i_{\alpha_m,H_m}$

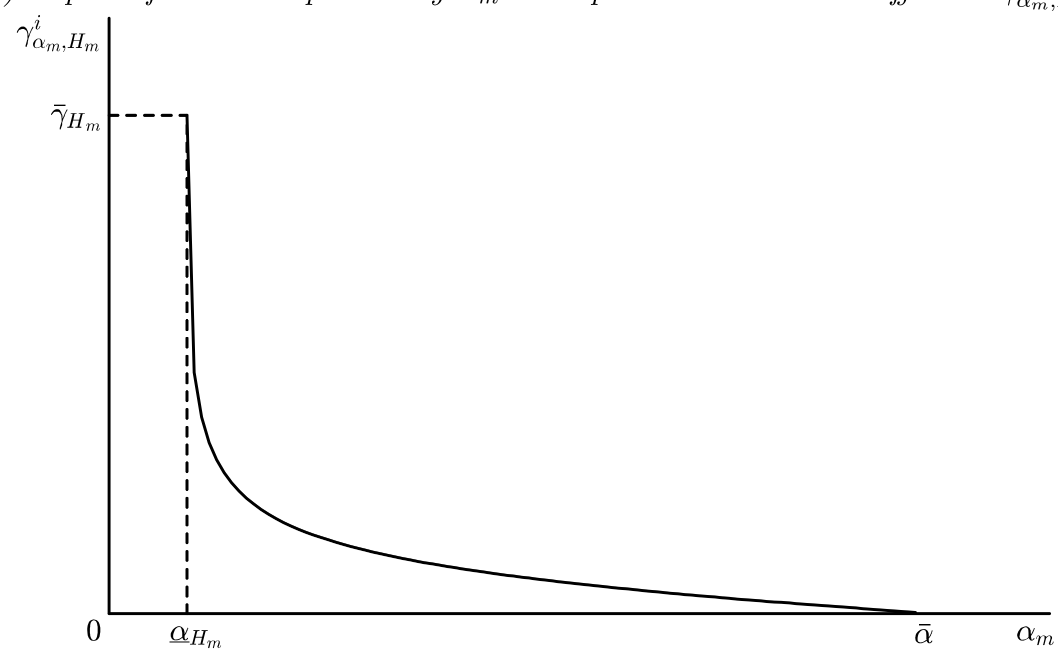


Fig. 9. The aggregate portfolio

The curve represents portfolios on the mean-variance frontier in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. Suppose that there are three accounts ($m = 1, 2, 3$). For any account $m \in \{1, 2, 3\}$, the threshold probability is $\alpha_m \in (0, \overline{\alpha})$, the threshold return is $H_m \in (-\infty, \overline{H}_{\alpha_m}]$, the optimal portfolio is $\boldsymbol{w}_{\alpha_m,H_m}$, and its expected return is E_{α_m,H_m}. The figure assumes that $\alpha_3 > \alpha_2 > \alpha_1$ and $H_3 < H_2 < H_1$. Given a vector of fractions of wealth in the accounts $\boldsymbol{y} \in \mathbb{R}^3_{++}$, the aggregate portfolio is $\boldsymbol{w}_a = \sum_{m=1}^{3} y_m \boldsymbol{w}_{\alpha_m,H_m}$ and its expected return is $E_a = \sum_{m=1}^{3} y_m E_{\alpha_m,H_m}$. The first, second, third, and fourth dots ('•') from the left plot, respectively, $\boldsymbol{w}_{\alpha_1,H_1}$, $\boldsymbol{w}_{\alpha_2,H_2}$, $\boldsymbol{w}_a$, and $\boldsymbol{w}_{\alpha_3,H_3}$. For any account $m \in \{1, 2, 3\}$, $\boldsymbol{w}_{\alpha_m,H_m}$ lies at the point where a line with intercept H_m and slope z_{α_m} crosses the top half of the frontier. Since the figure assumes that $\alpha_3 > \alpha_2 > \alpha_1$ and $H_3 < H_2 < H_1$, probability constraint (16) loosens when moving from account $m = 1$ to account $m = 2$ and in turn to account $m = 3$. Hence, $\boldsymbol{w}_{\alpha_3,H_3}$ lies above $\boldsymbol{w}_{\alpha_2,H_2}$, which in turn lies above $\boldsymbol{w}_{\alpha_1,H_1}$.

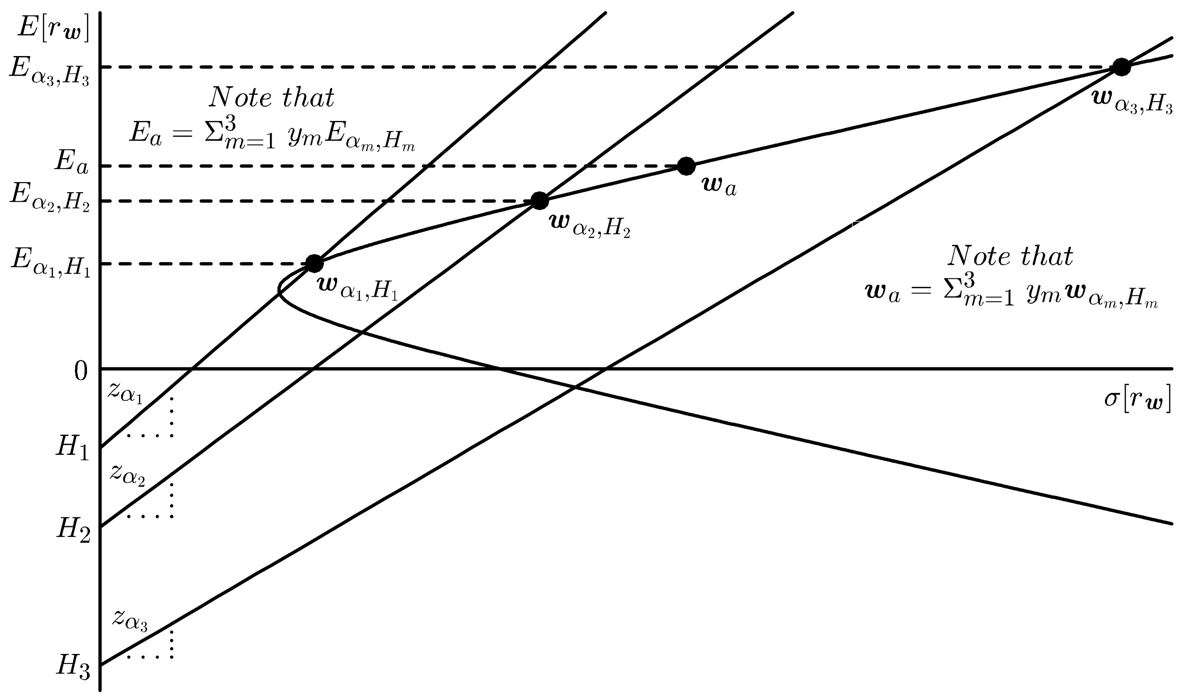


Fig. 10. Loss in CER arising from using an erroneous risk aversion coefficient

An investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$ who uses an erroneous risk aversion coefficient of $\gamma_\varepsilon \in \mathbb{R}_{++}$ in seeking to find the optimal portfolio has a loss in CER of $L_{\gamma,\gamma_\varepsilon}$; see Eq. (9). The thin dashed, thick dashed, thin solid, thick solid curves show how $L_{\gamma,\gamma_\varepsilon}$ depends on $\gamma_\varepsilon \in (0, 10]$ for, respectively, investors 1, 2, 3, and 4. These curves are truncated so that $L_{\gamma,\gamma_\varepsilon}$ is 10% or less. Recall that investors 1, 2, 3, and 4 have risk aversion coefficients of, respectively, 3.7950, 2.7063, 0.8773, and 2.1740; see the last column of Table 2(b).

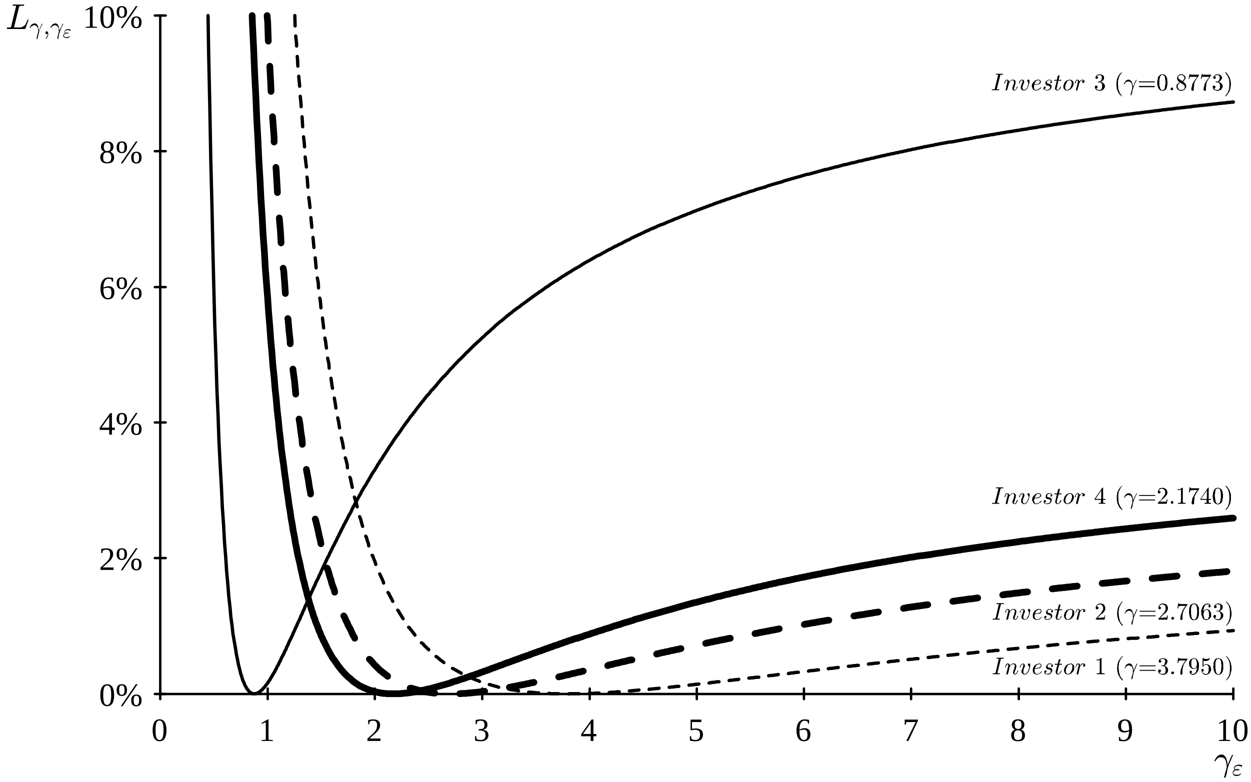


Fig. 11. Existence of optimal portfolios within accounts

The dashed vertical line goes through $\alpha_m = \overline{\alpha} = 33.40\%$; see Eq. (20). The curve plots $\overline{H}_{\alpha_m}$ as a function of $\alpha_m \in (0, \overline{\alpha})$; see Eq. (21). This curve is truncated so that $\overline{H}_{\alpha_m}$ is -16% or more. The dots ('•') plot the pairs of thresholds of accounts 1, 2, and 3, $\{(\alpha_m, H_m)\}_{m=1}^{3}$; see the last two columns of Table 2(c). For any account $m \in \{1, 2, 3\}$, note that the pair of thresholds (α_m, H_m) plots *both*: (a) strictly between the y-axis and the dashed vertical line so that $\alpha_m \in (0, \overline{\alpha})$; and (b) below the curve so that $H_m \in (-\infty, \overline{H}_{\alpha_m})$. Hence, optimal portfolios within accounts exist; see Theorem 5(i).

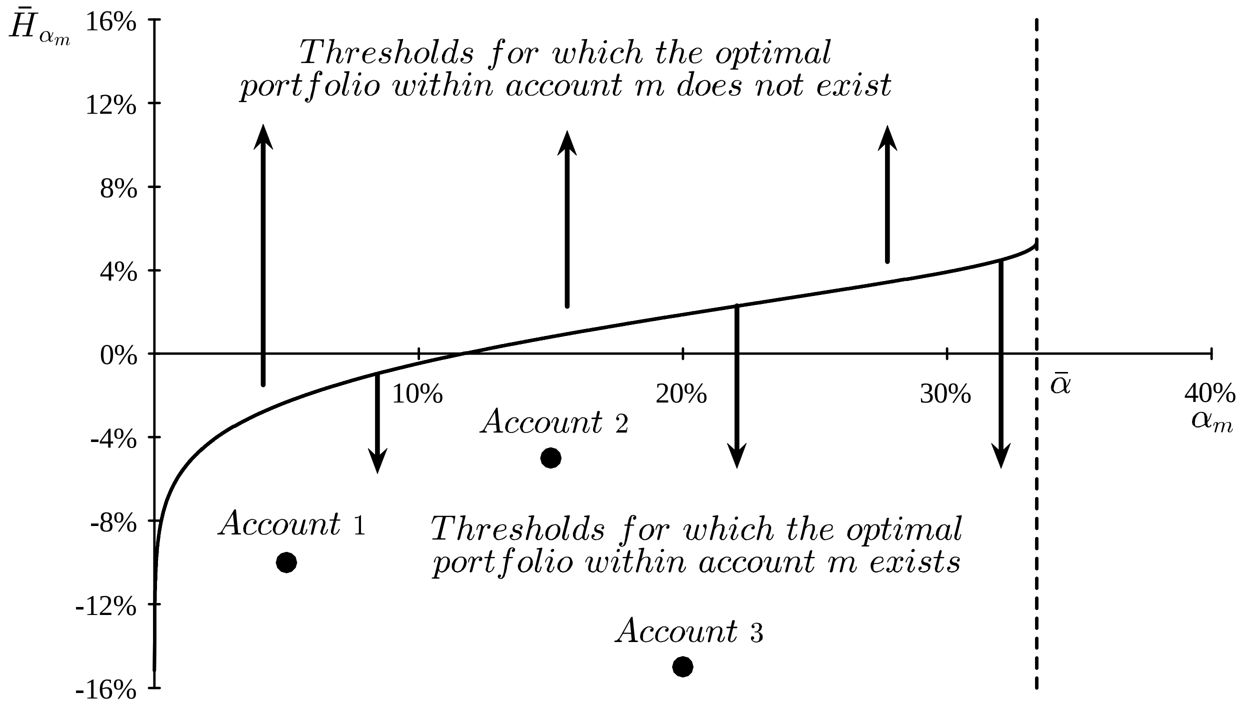


Fig. 12. Thresholds of three accounts with the same optimal portfolios as accounts 1, 2, and 3

The thin dashed, thick dashed, and solid curves identify pairs of thresholds $\{(\alpha, H_\alpha)\}_{\alpha\in(0,\overline{\overline{\alpha}})}$ of three accounts with the same optimal portfolios as, respectively, accounts 1, 2, and 3; see Eqs. (31) and (32) as well as Theorem 6. The solid curve is truncated so that H_α is -100% or more. The dots ('•') on the thin dashed, thick dashed, and solid curves plot the pairs of thresholds of, respectively, accounts 1, 2, and 3; see the last two columns of Table 2(c).

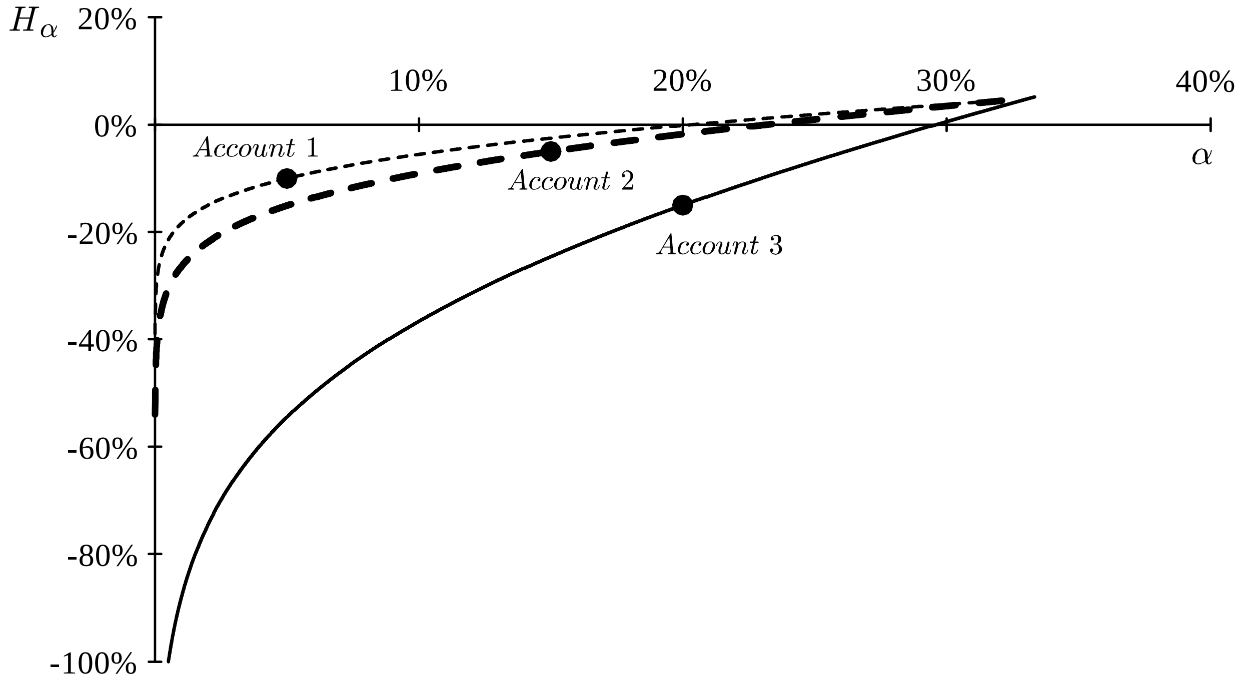


Fig. 13. Impact of the thresholds on the risk aversion coefficients implied by the optimal portfolios within accounts 1, 2, and 3

Given a threshold probability $\alpha_1 = 5\%$, Fig. 13(a) plots the risk aversion coefficient implied by the optimal portfolio within account 1, $\gamma^i_{\alpha_1,H_1}$, as a function of threshold return $H_1 \in [-40\%, \overline{H}_{\alpha_1}]$; see Eq. (21). Given $H_1 = -10\%$, Fig. 13(b) plots $\gamma^i_{\alpha_1,H_1}$ as a function of $\alpha_1 \in [\underline{\alpha}_{H_1}, 20\%]$; see Eq. (30). Given $\alpha_2 = 15\%$, Fig. 13(c) plots $\gamma^i_{\alpha_2,H_2}$ as a function of $H_2 \in [-40\%, \overline{H}_{\alpha_2}]$. Given $H_2 = -5\%$, Fig. 13(d) plots $\gamma^i_{\alpha_2,H_2}$ as a function of $\alpha_2 \in [\underline{\alpha}_{H_2}, 20\%]$. Given $\alpha_3 = 20\%$, Fig. 13(e) plots $\gamma^i_{\alpha_3,H_3}$ as a function of $H_3 \in [-40\%, \overline{H}_{\alpha_3}]$. Given $H_3 = -15\%$, Fig. 13(f) plots $\gamma^i_{\alpha_3,H_3}$ as a function of $\alpha_3 \in [\underline{\alpha}_{H_3}, 20\%]$.

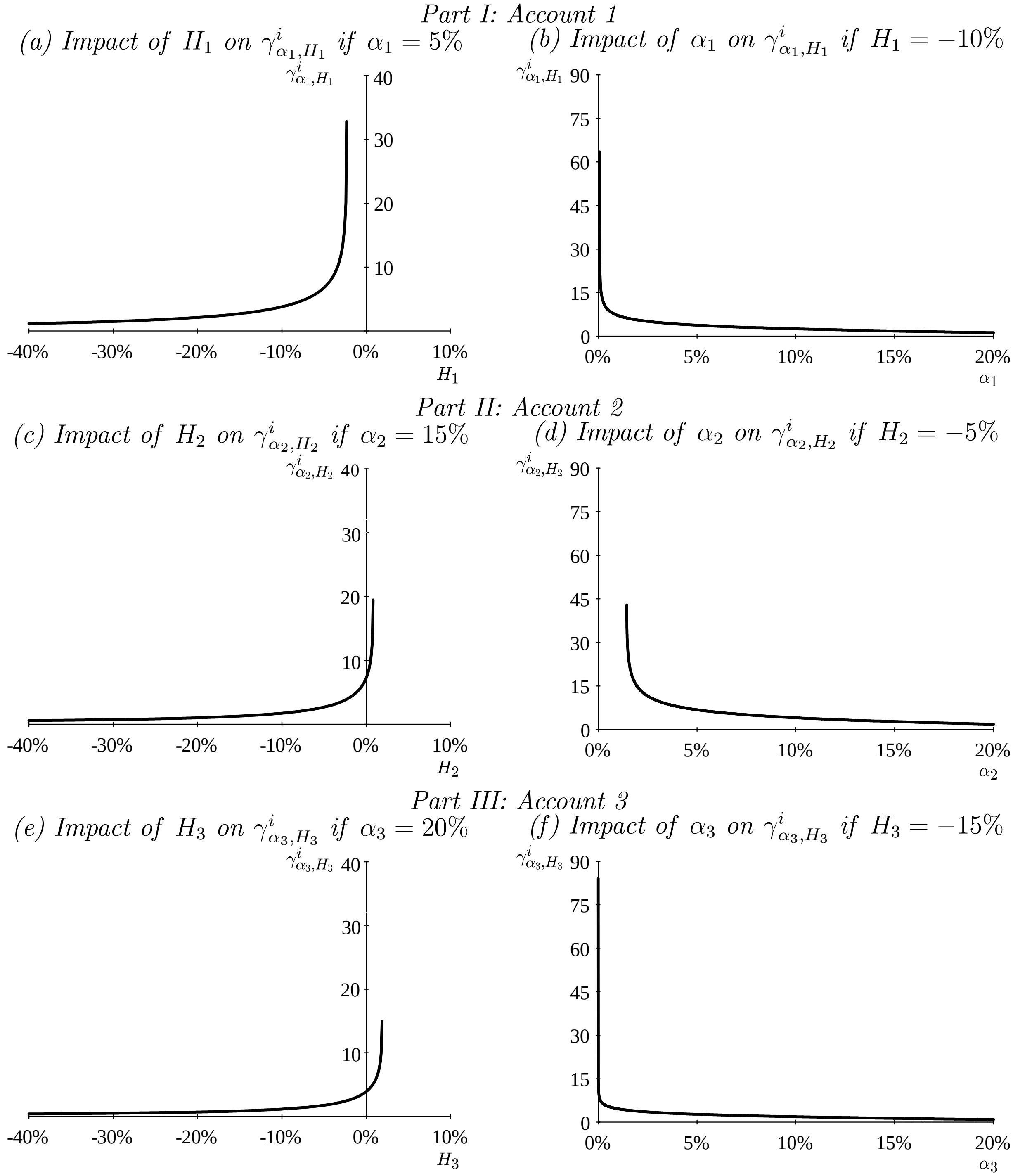


Table 1. Differences between the models of Markowitz and Das et al.

This table lists six crucial differences between the models of Markowitz and Das et al. The leftmost column identifies dimensions along which these models differ. The middle and rightmost columns show the features of the models of, respectively, Markowitz and Das et al.

Dimension along which models differ	*Model*	
	Markowitz	*Das et al.*
1. Number of accounts	One	Two or more
2. Parameters used to specify preferences	Risk aversion coefficient	Thresholds
3. Existence of the optimal portfolio	The optimal portfolio always exists	The optimal portfolios within accounts and the aggregate portfolio may or may not exist depending on the thresholds and optimization inputs
4. Location of the optimal portfolio along mean-variance frontier	The optimal portfolio (of an investor with a positive risk aversion coefficient) lies on the mean-variance frontier above the global minimum-variance portfolio	The optimal portfolio within an account (of an investor with thresholds such that the portfolio exists) lies on the mean-variance frontier at the same point as or above the global mininum-VaR portfolio at a confidence level equal to one minus the account's threshold probability
5. Mapping between parameters used to specify preferences and the optimal portfolio	Different risk aversion coefficients lead to different optimal portfolios	Infinitely many pairs of thresholds lead to the same optimal portfolio within an account
6. Risk aversion coefficient (or implied risk aversion coefficient) and the impact of the optimization inputs on this coefficient	The risk aversion coefficient of an investor is unique and does not depend on the optimization inputs	The risk aversion coefficients implied by optimal portfolios within accounts of an investor possibly differ across accounts and depend on the optimization inputs; the risk aversion coefficient implied by the aggregate portfolio of the investor also depends on the optimization inputs

Table 2. Optimization inputs and values of the parameters used to specify the preferences of investors in the models of Markowitz and Das et al.

Table 2(a) contains the optimization inputs for three assets (1, 2, and 3). Table 2(b) shows the risk aversion coefficients of four investors (1, 2, 3, and 4) in Markowitz's model. Table 2(c) shows the fractions of wealth and thresholds of three accounts (1, 2, and 3) of an investor in Das et al.'s model. The values in Tables 2(a), 2(b), and 2(c) are the same as the values used in the numerical example of Das et al.

(a) Optimization inputs

Asset	Expected return	Standard deviation	Correlation coefficient		
			Asset 1	Asset 2	Asset 3
1 (risky bond)	5%	5%	1.0	0.0	0.0
2 (low-risk stock)	10%	20%		1.0	0.2
3 (high-risk stock)	25%	50%			1.0

(b) Risk aversion coefficients of investors in Markowitz's model

Investor	Risk aversion coefficient
1	3.7950
2	2.7063
3	0.8773
4	2.1740

(c) Fractions of wealth and thresholds of the accounts of an investor in Das et al.'s model

Account	Fraction of wealth	Threshold probability	Threshold return
1	60%	5%	−10%
2	20%	15%	−5%
3	20%	20%	−15%

Table 3. Optimal portfolios for investors in the models of Markowitz and Das et al.

Table 3(a) provides the asset weights, expected returns, standard deviations of the optimal portfolios of investors 1, 2, 3, and 4 in Markowitz's model; see the last column of Table 2(b) for their risk aversion coefficients. The first three rows of Table 3(b) provide the asset weights, expected returns, standard deviations, and VaRs of the optimal portfolios within accounts 1, 2, and 3 of an investor in Das et al.'s model as well as the risk aversion coefficients implied by these portfolios; see the last two columns of Table 2(c) for the thresholds. The VaR of the optimal portfolio within a given account is reported at a confidence level equal to one minus the account's threshold probability. The last row of Table 3(b) provides the asset weights, expected return, and standard deviation of the aggregate portfolio of the investor in Das et al.'s model (the combination of the investor's optimal portfolios within accounts) as well as the risk aversion coefficient implied by this portfolio; see the second column of Table 2(c) for the fractions of wealth in the accounts.

(a) Optimal portfolios of investors in Markowitz's model

	Optimal portfolio				
	Weights			Expected	Standard
Investor	Asset 1	Asset 2	Asset 3	Return	Deviation
1	53.94%	26.56%	19.49%	10.23%	12.30%
2	37.87%	34.99%	27.14%	12.17%	16.57%
3	−78.90%	96.19%	82.70%	26.35%	49.13%
4	24.16%	42.17%	33.67%	13.84%	20.32%

(b) The optimal portfolios within accounts and the aggregate portfolio of an investor in Das et al.'s model

	Optimal portfolios within accounts and aggregate portfolio						
	Weights			Expected	Standard		Implied risk
Account	Asset 1	Asset 2	Asset 3	Return	Deviation	VaR	aversion coefficient
1	53.94%	26.56%	19.49%	10.23%	12.30%	10%	3.7950
2	37.87%	34.99%	27.14%	12.17%	16.57%	5%	2.7063
3	−78.90%	96.19%	82.70%	26.35%	49.13%	15%	0.8773
Aggregate portfolio	24.16%	42.17%	33.67%	13.84%	20.32%	–	2.1740

Online Appendix

Table A1. Summary of notation

Table A1(a) summarizes the notation used for Markowitz's model, which is also used for the Das et al.'s model. Table A1(b) summarizes additional notation used for the latter model.

(a) Notation used for Markowitz's model

N	Number of assets
$\boldsymbol{\mu}$	$N \times 1$ vector of expected asset returns
$\boldsymbol{\Sigma}$	$N \times N$ variance-covariance matrix for asset returns
$\boldsymbol{w}$	$N \times 1$ vector of asset weights in a portfolio
$r_{\boldsymbol{w}}$	Random return of portfolio $\boldsymbol{w}$
$E[r_{\boldsymbol{w}}]$	Mean or expected return of portfolio $\boldsymbol{w}$
$\sigma^2[r_{\boldsymbol{w}}]$	Variance of portfolio $\boldsymbol{w}$
$\sigma[r_{\boldsymbol{w}}]$	Standard deviation of portfolio $\boldsymbol{w}$
A, B, C, D	Constants used to analytically characterize the mean-variance frontier
$\boldsymbol{w}_0$	Global minimum-variance portfolio
$\boldsymbol{w}_1$	Portfolio located in $(E[r_{\boldsymbol{w}}], \sigma^2[r_{\boldsymbol{w}}])$ where a ray from the origin that goes through portfolio $\boldsymbol{w}_0$ crosses the mean-variance frontier
$\boldsymbol{w}_E$	Portfolio on the mean-variance frontier with an expected return of E
$\theta_E, 1-\theta_E$	Weights of portfolios $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ in portfolio $\boldsymbol{w}_E$
γ	Risk aversion coefficient
$\boldsymbol{w}_\gamma$	Optimal portfolio with a risk aversion coefficient of γ
$\theta_\gamma, 1-\theta_\gamma$	Weights of portfolios $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ in portfolio $\boldsymbol{w}_\gamma$
E_γ, σ_γ	Expected return and standard deviation of portfolio $\boldsymbol{w}_\gamma$
γ_ε	Erroneous risk aversion coefficient
$L_{\gamma,\gamma_\varepsilon}$	Loss in CER arising from an investor using γ_ε instead of γ
$\gamma_{\varepsilon,\kappa-}, \gamma_{\varepsilon,\kappa^+}$	Erroneous risk aversion coefficients, respectively, below and above γ by percentage κ
κ	Percentage that $\gamma_{\varepsilon,\kappa-}$ and $\gamma_{\varepsilon,\kappa^+}$ are, respectively, below and above γ
$L_{\gamma,\gamma_{\varepsilon,\kappa-}}, L_{\gamma,\gamma_{\varepsilon,\kappa+}}$	Losses in CER for an investor using, respectively, $\gamma_{\varepsilon,\kappa-}$ and $\gamma_{\varepsilon,\kappa^+}$ instead of γ
RCL_κ	Relative change in the loss in CER for an investor using $\gamma_{\varepsilon,\kappa-}$ instead of $\gamma_{\varepsilon,\kappa+}$

(b) Additional notation used for Das et al.'s model

M	Number of accounts
$\boldsymbol{y}$	$M \times 1$ vector of fractions of wealth allocated to the accounts
α_m, H_m	Threshold probability and threshold return of account m
$1-\alpha$	Confidence level to compute VaR
$\Phi(\cdot)$	Cumulative univariate standard Normal distribution function
z_α	Minus the inverse of $\Phi(\cdot)$ evaluated at α
$V[1-\alpha, r_{\boldsymbol{w}}]$	VaR at confidence level $1-\alpha$ of portfolio $\boldsymbol{w}$
$\overline{\alpha}$	Threshold probability at or above which optimal portfolios within accounts do not exist
$\overline{H}_\alpha$	Threshold return at or below which the optimal portfolio within a given account exists when threshold probability $\alpha \in (0, \overline{\alpha})$
$\boldsymbol{w}_{1-\alpha}$	Global minimum-VaR portfolio at confidence level $1-\alpha$
$\theta_{1-\alpha}, 1-\theta_{1-\alpha}$	Weights of portfolios $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ in portfolio $\boldsymbol{w}_{1-\alpha}$
$\boldsymbol{w}_{\alpha_m,H_m}$	Optimal portfolio within account m
$\theta_{\alpha_m,H_m}, 1-\theta_{\alpha_m,H_m}$	Weights of portfolios $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ in portfolio $\boldsymbol{w}_{\alpha_m,H_m}$
$E_{\alpha_m,H_m}, \sigma_{\alpha_m,H_m}$	Expected return and standard deviation of portfolio $\boldsymbol{w}_{\alpha_m,H_m}$
V_{α_m,H_m}	VaR at confidence level $1-\alpha_m$ of portfolio $\boldsymbol{w}_{\alpha_m,H_m}$
$\gamma^i_{\alpha_m,H_m}$	Risk aversion coefficient implied by portfolio $\boldsymbol{w}_{\alpha_m,H_m}$
α, H_α	Thresholds of account with optimal portfolio equal to $\boldsymbol{w}_{\alpha_m,H_m}$
$1-\overline{\overline{\alpha}}$	Confidence level at which $\boldsymbol{w}_{\alpha_m,H_m}$ equals $\boldsymbol{w}_{1-\overline{\overline{\alpha}}}$
$\overline{\gamma}_{\alpha_m}$	Risk aversion coefficient implied by portfolio $\boldsymbol{w}_{\alpha_m,\overline{H}_{\alpha_m}}$
$\underline{\alpha}_{H_m}$	Threshold probability below which the optimal portfolio within account m does not exist when threshold return $H_m \in (-\infty, A/C)$
$\overline{\gamma}_{H_m}$	Risk aversion coefficient implied by portfolio $\boldsymbol{w}_{\underline{\alpha}_{H_m},H_m}$
$\alpha_\gamma,\ H_\gamma$	Thresholds of account with an optimal portfolio equal to $\boldsymbol{w}_\gamma$
$\boldsymbol{w}_a$	Aggregate portfolio
$\theta_a, 1-\theta_a$	Weights of portfolios $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ in portfolio $\boldsymbol{w}_a$
E_a, σ_a	Expected return and standard deviation of portfolio $\boldsymbol{w}_a$
γ^i_a	Risk aversion coefficient implied by portfolio $\boldsymbol{w}_a$

Proofs of theoretical results

Proof of Theorem 1. Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$. Using Eq. (2), the following holds:

$$\sigma^2[r_{\boldsymbol{w}}] = 1/C + (E[r_{\boldsymbol{w}}] - A/C)^2/(D/C) \tag{A.1}$$

for any portfolio $\boldsymbol{w}$ on the mean-variance frontier. Since $\boldsymbol{w}_\gamma$ is on the top half of the mean-variance frontier, Eq. (A.1) implies that E_γ solves:

$$\max_{E \in \mathbb{R}} \quad E - (\gamma/2)\left[1/C + (E - A/C)^2/(D/C)\right]. \tag{A.2}$$

A first-order condition for E_γ to solve maximization problem (A.2) is $1 - \gamma \frac{E_\gamma - A/C}{D/C} = 0$. Therefore, Eq. (7) holds. Using Eq. (1) as well as the facts that $\boldsymbol{w}_\gamma$ is on the mean-variance frontier and has an expected return of E_γ, Eq. (6) holds. Eq. (8) follows from Eqs. (7) and (A.1). ■

Proof of Corollary 1. First, consider part (i). Fix an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$. Since $D/C > 0$ and $\gamma > 0$, Eq. (7) implies that $E_\gamma > A/C$. Recall that $\boldsymbol{w}_0$ has an expected return of A/C. Hence, $\boldsymbol{w}_\gamma$ lies above $\boldsymbol{w}_0$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. This completes the proof of part (i).

Second, consider part (ii). Eq. (7) implies that E_γ converges to A/C and thus θ_γ converges to one as γ converges to infinity. It follows from Eq. (6) that $\boldsymbol{w}_\gamma$ converges to $\boldsymbol{w}_0$ as γ converges to infinity. This completes the proof of part (ii). ■

Proof of Corollary 2. Since $D/C > 0$, Eq. (7) implies that E_γ converges to infinity as γ converges to zero. ■

Proof of Theorem 2. Consider an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$ who uses an erroneous risk aversion coefficient of $\gamma_\varepsilon \in \mathbb{R}_{++}\backslash\{\gamma\}$ in seeking to find the optimal portfolio. It follows from Eqs. (3), (7), and (8) that:

$$U(E[r_{\boldsymbol{w}_\gamma}], \sigma[r_{\boldsymbol{w}_\gamma}]) = A/C + (D/C)/(2\gamma) - (\gamma/C)/2. \tag{A.3}$$

Using Eqs. (7) and (8) with $\gamma = \gamma_\varepsilon$:

$$E_{\gamma_\varepsilon} = A/C + (D/C)/\gamma_\varepsilon \tag{A.4}$$

and:

$$\sigma_{\gamma_\varepsilon} = \sqrt{1/C + (D/C)/\gamma_\varepsilon^2}. \tag{A.5}$$

It follows from Eqs. (A.4) and (A.5) that:

$$U(E[r_{\boldsymbol{w}_{\gamma_\varepsilon}}], \sigma[r_{\boldsymbol{w}_{\gamma_\varepsilon}}]) = A/C + (D/C)/\gamma_\varepsilon - (\gamma/C)/2 - (\gamma/2)(D/C)/\gamma_\varepsilon^2. \tag{A.6}$$

Using Eqs. (A.3) and (A.6):

$$U(E[r_{\boldsymbol{w}_\gamma}], \sigma[r_{\boldsymbol{w}_\gamma}]) - U(E[r_{\boldsymbol{w}_{\gamma_\varepsilon}}], \sigma[r_{\boldsymbol{w}_{\gamma_\varepsilon}}]) = (D/C)/(2\gamma) - (D/C)/\gamma_\varepsilon + (\gamma/2)(D/C)/\gamma_\varepsilon^2. \tag{A.7}$$

Eq. (10) follows from Eqs. (9) and (A.7) as well as elementary algebra. ■

Proof of Theorem 3. Fix an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$ who uses an erroneous risk aversion coefficient of $\gamma_\varepsilon \in \mathbb{R}_{++} \backslash \{\gamma\}$ in seeking to find the optimal portfolio. First, consider part (i). Suppose that $\gamma_\varepsilon = \gamma_{\varepsilon,\kappa^-} = (1-\kappa)\gamma$ where $\kappa \in (0,1)$. Since $\gamma_\varepsilon = (1-\kappa)\gamma$, Eq. (11) follows from Eq. (10). This completes the proof of part (i).

Second, consider part (ii). Suppose that $\gamma_\varepsilon = \gamma_{\varepsilon,\kappa^+} = (1+\kappa)\gamma$ where $\kappa \in (0,\infty)$. Since $\gamma_\varepsilon = (1+\kappa)\gamma$, Eq. (12) follows from Eq. (10). This completes the proof of part (ii).

Third, consider part (iii). Suppose that $\kappa \in (0,1)$. Eq. (13) follows from Eqs. (11) and (12). This completes the proof of part (iii). ■

Proof of Theorem 4. The proof is similar to the Proof of Proposition 1 in Alexander and Baptista (2002). In particular, Eqs. (23), (24), and (25) follows from, respectively, Eqs. (A.8), (A.6), and (11) in Alexander and Baptista. ■

Proof of Theorem 5. The proof of follows by replacing symbols with a superscript 'ε' with symbols without this superscript in the Proof of Theorem 1 in the Online Appendix of Alexander et al. (2017).[A.1] ■

Proof of Corollary 3. Fix an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$. It is convenient to first show part (ii). Suppose that $H_m = \overline{H}_{\alpha_m}$. Using Eq. (21) with $\alpha = \alpha_m$:

$$\overline{H}_{\alpha_m} = A/C - \sqrt{(z_{\alpha_m}^2 - D/C)/C}. \tag{A.8}$$

Substituting H_m in the right-hand side of Eq. (28) with the right-hand side of Eq. (A.8) and using elementary algebra:

$$\sigma_{\alpha_m,H_m} = \sqrt{\left(z_{\alpha_m}^2/C\right)/(z_{\alpha_m}^2 - D/C)}. \tag{A.9}$$

[A.1] See Alexander, G. J., A. M. Baptista, and S. Yan. 2017. "Portfolio Selection with Mental Accounts and Estimation Risk." *Journal of Empirical Finance* 41, 161–186. doi: 10.1016/j.jempfin.2016.07.012. The Online Appendix of Alexander et al. (2017) is available at: <blogs.gwu.edu/alexbapt/files/2017/03/JEFAppendix-2m65ivr.pdf>. In Alexander et al., the symbols with superscript 'ε' are associated to the use of estimated optimization inputs $\boldsymbol{\mu}^\varepsilon$ and $\boldsymbol{\Sigma}^\varepsilon$ (instead of the 'true' optimization inputs $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$). In comparison, the subscript 'ε' in 'γ_ε' is used here to denote the erroneous risk aversion coefficient of an investor in Markowitz's model.

Eqs. (24) and (A.9) imply that $\sigma_{\alpha_m,H_m} = \sigma_{1-\alpha_m}$. Since $\boldsymbol{w}_{\alpha_m,H_m}$ and $\boldsymbol{w}_{1-\alpha_m}$ lie on the top half of the frontier, $\boldsymbol{w}_{\alpha_m,H_m} = \boldsymbol{w}_{1-\alpha_m}$. This completes the proof of part (ii).

Next, consider part (i). Suppose that $H_m \in (-\infty, \overline{H}_{\alpha_m})$. Since $H_m < \overline{H}_{\alpha_m}$, Eq. (21) implies that:

$$H_m < A/C - (z_{\alpha_m}^2 - D/C)/C. \tag{A.10}$$

It follows from Eq. (A.10) that:

$$(z_{\alpha_m}^2 - D/C)/C < A/C - H_m. \tag{A.11}$$

Since $\alpha_m \in (0, \overline{\alpha})$, note that $z_{\alpha_m}^2 > D/C$. The fact that $z_{\alpha_m}^2 > D/C$ and Eq. (A.11) imply that $A/C - H_m > 0$. It follows from Eq. (28) that:

$$\frac{\partial \sigma_{\alpha_m,H_m}}{\partial H_m} = \frac{-z_{\alpha_m} + \frac{-D/C(A/C-H_m)}{\sqrt{(D/C)[(A/C-H_m)^2-(z_{\alpha_m}^2-D/C)/C]}}}{z_{\alpha_m}^2 - D/C}. \tag{A.12}$$

Since $z_{\alpha_m} > 0$, $D/C > 0$, and $A/C - H_m > 0$:

$$\frac{\partial \sigma_{\alpha_m,H_m}}{\partial H_m} < 0. \tag{A.13}$$

Using the assumption that $H_m < \overline{H}_{\alpha_m}$, Corollary 3(ii), and Eq. (A.13), $\sigma_{\alpha_m,H_m} > \sigma_{1-\alpha_m}$. Since $\boldsymbol{w}_{\alpha_m,H_m}$ and $\boldsymbol{w}_{1-\alpha_m}$ lie on the top half of the frontier $\boldsymbol{w}_{\alpha_m,H_m}$ lies above $\boldsymbol{w}_{1-\alpha_m}$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. This completes the proof of part (i). ■

Proof of Corollary 4. Fix an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$. Since $z_{\alpha_m}^2 > D/C > 0$, Eq. (28) implies that the σ_{α_m,H_m} converges to infinity as H_m converges to minus infinity. It follows from Eq. (27) that E_{α_m,H_m} also converges to infinity as H_m converges to minus infinity. ■

Proof of Corollary 5. Fix an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in [\underline{\alpha}_{H_m}, \overline{\alpha})$ and threshold return $H_m \in (-\infty, A/C)$. It is convenient to first show part (ii). Suppose that $\alpha_m = \underline{\alpha}_{H_m}$. It follows from Eq. (30) that:

$$z_{\alpha_m} = \sqrt{D/C + C(A/C - H_m)^2}. \tag{A.14}$$

Eqs. (21) and (A.14) imply that:

$$\overline{H}_{\alpha_m} = H_m. \tag{A.15}$$

Using Corollary 3(ii), $\boldsymbol{w}_{\alpha_m,H_m} = \boldsymbol{w}_{1-\alpha_m}$. This completes the proof of part (ii).

Consider now part (i). Suppose that $\alpha_m \in (\underline{\alpha}_{H_m}, \overline{\alpha})$. For brevity, let:

$$F \equiv \sqrt{(D/C)[(A/C - H_m)^2 - (z_{\alpha_m}^2 - D/C)/C]}. \tag{A.16}$$

Eqs. (28) and (A.16) imply that:

$$\frac{\partial \sigma_{\alpha_m,H_m}}{\partial z_{\alpha_m}} = \frac{\left[(A/C - H_m) + \frac{-\left(D/C^2\right) z_{\alpha_m}}{F}\right]\left(z_{\alpha_m}^2 - D/C\right) - 2z_{\alpha_m}\left[z_{\alpha_m}\left(A/C - H_m\right) + F\right]}{\left(z_{\alpha_m}^2 - D/C\right)^2}. \tag{A.17}$$

It follows from Eq. (A.17) that:

$$\frac{\partial \sigma_{\alpha_m,H_m}}{\partial z_{\alpha_m}} = \frac{\left[(A/C - H_m) F - \left(D/C^2\right) z_{\alpha_m}\right]\left(z_{\alpha_m}^2 - D/C\right) - 2z_{\alpha_m}\left[z_{\alpha_m}\left(A/C - H_m\right) + F\right] F}{\left(z_{\alpha_m}^2 - D/C\right)^2 F}. \tag{A.18}$$

Simplifying Eq. (A.18):

$$\frac{\partial \sigma_{\alpha_m,H_m}}{\partial z_{\alpha_m}} = \frac{-\left(A/C - H_m\right) F \left(D/C\right) - z_{\alpha_m}\left(D/C^2\right)\left(z_{\alpha_m}^2 - D/C\right) - z_{\alpha_m}^2\left(A/C - H_m\right) F - 2z_{\alpha_m} F^2}{\left(z_{\alpha_m}^2 - D/C\right)^2 F}. \tag{A.19}$$

Since $z_{\alpha_m} > \sqrt{D/C} > 0$, $A/C - H_m > 0$, and $F > 0$, it follows from Eq. (A.19) that:

$$\frac{\partial \sigma_{\alpha_m,H_m}}{\partial z_{\alpha_m}} < 0. \tag{A.20}$$

Noting that $\frac{\partial z_{\alpha_m}}{\partial \alpha_m} < 0$, Eq. (A.20) implies that:

$$\frac{\partial \sigma_{\alpha_m,H_m}}{\partial \alpha_m} > 0. \tag{A.21}$$

Using the fact that $\alpha_m > \underline{\alpha}_{H_m}$, Corollary 5(ii), and Eq. (A.21), $\sigma_{\alpha_m,H_m} > \sigma_{1-\alpha_m}$. Since $\boldsymbol{w}_{\alpha_m,H_m}$ and $\boldsymbol{w}_{1-\alpha_m}$ lie on the top half of the frontier, $\boldsymbol{w}_{\alpha_m,H_m}$ lies above $\boldsymbol{w}_{1-\alpha_m}$ in $(E[r_{\boldsymbol{w}}], \sigma[r_{\boldsymbol{w}}])$ space. This completes the proof of part (i). ■

Proof of Corollary 6. Fix an account $m \in \{1, ..., M\}$ with threshold return $H_m \in (-\infty, A/C)$. Using the definition of z_{α_m} and Eq. (20), z_{α_m} converges to $\sqrt{D/C}$ as α_m converges to $\overline{\alpha}$ from below. Therefore, the assumption that $H_m < A/C$ and Eq. (28) imply that σ_{α_m,H_m} converges to infinity as α_m converges to $\overline{\alpha}$ from below. It follows from Eq. (27) that E_{α_m,H_m} also converges to infinity as α_m converges to $\overline{\alpha}$ from below. ■

Proof of Theorem 6. Fix an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$. Fix another account with threshold probability $\alpha \in (0, \overline{\overline{\alpha}}]$ and threshold return H_α. Eqs. (27) and (32) imply that:

$$A/C - H_\alpha = z_\alpha \sigma_{\alpha_m,H_m} - \sqrt{(D/C)(\sigma_{\alpha_m,H_m}^2 - 1/C)}. \tag{A.22}$$

Using Eq. (28) with $(\alpha_m, H_m) = (\alpha, H_\alpha)$ as well as Eq. (A.22):

$$\sigma_{\alpha,H_\alpha} = \frac{z_\alpha\left[z_\alpha\sigma_{\alpha_m,H_m} - \sqrt{(D/C)(\sigma^2_{\alpha_m,H_m} - 1/C)}\right]}{z^2_\alpha - D/C} + \frac{\sqrt{(D/C)\left\{\left[z_\alpha\sigma_{\alpha_m,H_m} - \sqrt{(D/C)(\sigma^2_{\alpha_m,H_m} - 1/C)}\right]^2 - (z^2_\alpha - D/C)/C\right\}}}{z^2_\alpha - D/C}. \tag{A.23}$$

It follows from Eq. (A.23) and elementary algebra that:

$$\sigma_{\alpha,H_\alpha} = \frac{z_\alpha\left[z_\alpha\sigma_{\alpha_m,H_m} - \sqrt{(D/C)(\sigma^2_{\alpha_m,H_m} - 1/C)}\right]}{z^2_\alpha - D/C} + \frac{\sqrt{(D/C)\left(z_\alpha\sqrt{\sigma^2_{\alpha_m,H_m} - 1/C} - \sqrt{D/C}\sigma_{\alpha_m,H_m}\right)^2}}{z^2_\alpha - D/C}. \tag{A.24}$$

Simplifying Eq. (A.24), $\sigma_{\alpha,H_\alpha} = \sigma_{\alpha_m,H_m}$. Since $\boldsymbol{w}_{\alpha,H_\alpha}$ and $\boldsymbol{w}_{\alpha_m,H_m}$ lie on the top half of the mean-variance frontier, $\boldsymbol{w}_{\alpha,H_\alpha} = \boldsymbol{w}_{\alpha_m,H_m}$. ■

Proof of Theorem 7. Fix an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$. Using Eq. (7) with $\gamma = (D/C)\,/\,(E_{\alpha_m,H_m} - A/C)$, $E_\gamma = E_{\alpha_m,H_m}$. Hence, Eq. (35) holds. ■

Proof of Corollary 7. Fix an account $m \in \{1, ..., M\}$ with threshold probability $\alpha_m \in (0, \overline{\alpha})$. First, consider part (i). It follows from Corollary 4 that E_{α_m,H_m} converges to infinity as H_m converges to minus infinity. Hence, Eq. (35) implies that $\gamma^i_{\alpha_m,H_m}$ converges to zero as H_m converges to minus infinity. This completes the proof of part (i).

Second, consider part (ii). Suppose that $H_m = \overline{H}_{\alpha_m}$. It follows from Corollary 3(ii) that $\boldsymbol{w}_{\alpha_m,H_m} = \boldsymbol{w}_{1-\alpha_m}$. Eqs. (23) and (35) imply that:

$$\gamma^i_{\alpha_m,H_m} = (D/C)\,/\sqrt{(D^2/C^3)/\left(z^2_{\alpha_m} - D/C\right)}. \tag{A.25}$$

Using Eq. (A.25), $\gamma^i_{\alpha_m,H_m} = \overline{\gamma}_{\alpha_m}$. This completes the proof of part (ii). ■

Proof of Corollary 8. Fix an account $m \in \{1, ..., M\}$ with threshold return $H_m \in (-\infty, A/C)$. First, consider part (i). Using Corollary 6, E_{α_m,H_m} converges to infinity as α_m to $\overline{\alpha}$ from below. Hence, Eq. (35) implies that $\gamma^i_{\alpha_m,H_m}$ converges to zero as α_m to $\overline{\alpha}$ from below. This completes the proof of part (i).

Second, consider part (ii). Suppose that $\alpha_m = \underline{\alpha}_{H_m}$. Using Corollary 5(ii), $\boldsymbol{w}_{\alpha_m,H_m} = \boldsymbol{w}_{1-\alpha_m}$. Since $\alpha_m = \underline{\alpha}_{H_m}$, it follows from Eq. (30) that:

$$z_{\alpha_m} = \sqrt{D/C + C(A/C - H_m)^2}. \tag{A.26}$$

Eqs. (23), (35), and (A.26) imply that:

$$\gamma^i_{\alpha_m,H_m} = (D/C)\,/\sqrt{(D^2/C^3)/[C(A/C - H_m)^2]}. \tag{A.27}$$

It follows from Eq. (A.27) that $\gamma^i_{\alpha_m,H_m} = \overline{\gamma}_{H_m}$. This completes the proof of part (ii). ■

Proof of Theorem 8. Fix an investor with a risk aversion coefficient of $\gamma \in \mathbb{R}_{++}$ as well as an account with threshold probability $\alpha_\gamma \in (0, \overline{\alpha}_\gamma]$ and threshold return of H_γ. It follows from Eqs. (2) and (7) that:

$$E_\gamma = A/C + \sqrt{(D/C)(\sigma^2_\gamma - 1/C)}. \tag{A.28}$$

Eqs. (A.28) and (39) imply that:

$$A/C - H_\gamma = z_{\alpha_\gamma}\sigma_\gamma - \sqrt{(D/C)(\sigma^2_\gamma - 1/C)} \tag{A.29}$$

Using Eq. (28) with $(\alpha_m, H_m) = (\alpha_\gamma, H_\gamma)$ as well as Eq. (A.29):

$$\begin{aligned}\sigma_{\alpha_\gamma,H_\gamma} \;=\; & \frac{z_{\alpha_\gamma}\left[z_{\alpha_\gamma}\sigma_\gamma - \sqrt{(D/C)(\sigma^2_\gamma - 1/C)}\right]}{z^2_{\alpha_\gamma} - D/C} \\ & + \frac{\sqrt{(D/C)\left\{\left[z_{\alpha_\gamma}\sigma_\gamma - \sqrt{(D/C)(\sigma^2_\gamma - 1/C)}\right]^2 - (z^2_{\alpha_\gamma} - D/C)/C\right\}}}{z^2_{\alpha_\gamma} - D/C}.\end{aligned} \tag{A.30}$$

It follows from Eq. (A.30) and elementary algebra that:

$$\begin{aligned}\sigma_{\alpha_\gamma,H_\gamma} \;=\; & \frac{z_{\alpha_\gamma}\left[z_{\alpha_\gamma}\sigma_\gamma - \sqrt{(D/C)(\sigma^2_\gamma - 1/C)}\right]}{z^2_{\alpha_\gamma} - D/C} \\ & + \frac{\sqrt{(D/C)\left(z_{\alpha_\gamma}\sqrt{\sigma^2_\gamma - 1/C} - \sqrt{D/C}\,\sigma_\gamma\right)^2}}{z^2_{\alpha_\gamma} - D/C}.\end{aligned} \tag{A.31}$$

Simplifying Eq. (A.31), $\sigma_{\alpha_\gamma,H_\gamma} = \sigma_\gamma$. Since $\boldsymbol{w}_{\alpha_\gamma,H_\gamma}$ and $\boldsymbol{w}_\gamma$ lie on the top half of the mean-variance frontier, $\boldsymbol{w}_{\alpha_\gamma,H_\gamma} = \boldsymbol{w}_\gamma$. ■

Proof of Theorem 9. Suppose that threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$ for any account $m \in \{1, ..., M\}$. Using Theorem 5(i), $\{\boldsymbol{w}_{\alpha_m,H_m}\}^M_{m=1}$ exist and so does $\boldsymbol{w}_a$. Eqs. (40) and (41) follow the definition of $\boldsymbol{w}_a$ as well as Eqs. (26) and (27). Using Eqs. (1) and (40), $\boldsymbol{w}_a$ is on the mean-variance frontier. Hence, Eq. (42) follows from Eqs. (2) and (41). ■

Proof of Theorem 10. Suppose that threshold probability $\alpha_m \in (0, \overline{\alpha})$ and threshold return $H_m \in (-\infty, \overline{H}_{\alpha_m}]$ for any account $m \in \{1, ..., M\}$. Using Eq. (7) with $\gamma = (D/C) \, / \, (E_a - A/C)$, $E_\gamma = E_a$. It follows that:

$$\gamma_a^i = (D/C) \, / \, (E_a - A/C) \, . \tag{A.32}$$

Since $\sum_{m=1}^{M} y_m = 1$ and $E_a = \sum_{m=1}^{M} y_m E_{\alpha_m, H_m}$, Eq. (35) implies that:

$$\sum_{m=1}^{M} y_m / \gamma_{\alpha_m, H_m}^i = (E_a - A/C) \, / \, (D/C) \, . \tag{A.33}$$

Eq. (45) follows from Eqs. (A.32) and (A.33). ■