Coordination Failures, Bank Runs and Asset Prices*

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Abstract

We introduce interbank asset markets into the Diamond and Dybvig (1983) banking model. This market allows a bank facing a bank run to sell its assets rather than physically liquidating them. The introduction of this market significantly alters the set of equilibria. In particular, equilibria exist in which the first-best allocation obtains and bank runs never occur. Moreover, for some parameter values, asymmetric equilibria exist in which some banks hold a portfolio that exposes them to a bank run, while other banks hold a portfolio that makes them immune to runs. Finally, the asset market is a new source of multiplicity of equilibria as beliefs about asset prices influence banks' portfolio choices, which in turn determine equilibrium asset prices. This multiplicity gives liquidity regulation a new role as a mechanism to select the most desirable equilibrium.

Keywords Banking · Interbank Asset Markets · Liquidity Insurance · Extrinsic Risk · Financial Stability

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1 Introduction

Sharp falls in asset prices are sometimes, but not always, accompanied by the failure of a significant number of financial intermediaries. In such instances, the drying-up of market liquidity often coincides with a sudden deterioration of the funding liquidity for some financial intermediaries, which then fail as a result of poor overall liquidity conditions. Aiming to promote the resilience of the banking sector to liquidity shocks, the Basel Committee has introduced global liquidity standards, effective as of 2018. They require banks to hold a minimum stock of liquid assets to withstand stressed funding scenarios, for example in form of massive deposit withdrawals or problems to roll-over existing short-term debt.

While the intentions of the regulator seem clear, there are still substantial limits to our understanding of the mechanisms through which the possibility of a liquidity crisis influences the interaction of asset prices and the portfolio choice of banks (Basel Committee on Banking Supervision, 2016). Such interaction is the focus in our paper, where we adapt the banking model of Diamond and Dybvig (1983) to include interbank asset markets. Banks serve as financial intermediaries that provide liquidity insurance for consumers. Liquidity shocks come in form of coordination failures, i.e. when a depositor withdraws her deposits from a bank only because she expects everyone else to do so. Such bank runs are triggered by an extrinsic random variable that is unrelated to the fundamentals, or sunspot for short. On secondary interbank asset markets, banks can trade reserves for long-term productive investments. Hence, in a bank run, a bank's productive investments can not only be unwound and physically liquidated but also sold. Interbank asset markets thus matter as to how a bank can transform its assets into consumption in a bank run.

Interbank asset markets also have an effect on a bank's portfolio choice between holding reserves and making productive investments. This is an important aspect of the interaction between banks and asset markets as it determines the susceptibility of a bank to bank runs. Specifically, the bank is immune to coordination failures provided the value of a bank's portfolio of reserves and productive investments allows the bank to meet the withdrawal demands of depositors independently of whether depositors run. A bank that is immune to runs is *run-proof*, otherwise a bank is *run-prone*.

The values of bank assets are equilibrium outcomes and potentially state-dependent. This endogeneity of asset values alters significantly the set and the characteristics of equilibria. For one, there are equilibria in which asset prices depend on the extrinsic state while bank runs do not occur and the first-best allocation obtains. Such equilibria with *trivial sunspots* exist provided the sunspot probability is not too large, and the range of probabilities supporting trivial sunspots converges to the unit interval if the physical liquidation value of the productive investments converges to zero.

The endogeneity of asset values is also a new source of *multiplicity of equilibria*, as beliefs about asset prices influence banks' portfolio choices, which in turn determine equilibrium asset prices. A necessary condition for equilibria with run-prone banks to exist is that beliefs are such that asset prices in the extrinsic state, in which consumers contemplate to run, will be lower than in the other state. For those prices, sunspots are necessarily non-trivial because banks never provide optimal liquidity insurance and equilibria feature either the occasional failure of (some) banks or real indeterminacy. If the sunspot probability is very low, only equilibria without run-proof banks (*risky* banking sector) exist, in which banks physically liquidate their productive investments in the sunspot state. If the sunspot probability is very high, only equilibria without run-prone banks (*safe* banking sector) exist, in which asset prices ensure that the value of run-proof banks' portfolios is sufficient to deter consumers from running. In general, if there is a safe banking sector, a range of asset prices supports such equilibrium. Asset prices can thus be indeterminate, implying real indeterminacy since allocations depend on asset prices in equilibria with non-trivial sunspots. If the sunspot probability is neither very high nor very low, asymmetric equilibria may exist where some banks are run-prone and others are run-proof (*mixed* banking sector). Having a mixture of bank types is what allows for an active asset market in equilibrium, where run-prone banks hold an illiquid portfolio and will be the sellers of productive investments, while run-proof banks hold a liquid portfolio and will be the buyers. Also for intermediate sunspot probabilities, there is potentially more than one equilibrium with non-trivial sunspots. These equilibria differ in the share of run-prone banks, asset prices and thus allocations.

Finally, the endogeneity of asset values affects the share banks allocate to productive investments. We show that with safe banking sectors, aggregate productive investments are larger than their efficient level, and with risky banking sectors, aggregate productive investments are smaller than their efficient level. The reason is that in the absence of run-proof banks, asset markets are illiquid. Banks have to physically liquidate production in system-wide bank runs. Hence, holding reserves is rather valuable in providing liquidity insurance. In equilibria without run-prone banks, asset markets are liquid. However, since banks offer less than the efficient liquidity insurance, fewer reserves are needed for providing this level of liquidity insurance. Therefore, there is a trade-off between market liquidity and bank reserves: while a safe banking sector is characterized by liquid asset markets and banks holding relatively small reserves, a risky banking sector features illiquid asset markets and banks with large reserves.

The possibility of multiple equilibria is a feature of our model that helps to understand the real economic implications of financial stability. The structure of the banking sector and asset prices are both equilibrium outcomes. Without multiplicity of equilibria, a necessary condition for a different banking sector and different asset prices would be that the fundamentals of the economy, such as preferences and technologies, or sunspot probabilities are different. Accordingly, differences in the real outcomes associated with different banking sectors and asset prices are, ultimately, due to differences in these characteristics of economies. Comparing equilibria from a set of multiple equilibria for a given economy, however, is like conducting a controlled experiment that allows to attribute any differences in real outcomes exclusively to differences in the financial sector. For example, while fluctuations of asset prices unrelated to fundamentals can be without adverse real economic consequences if the economy is in an equilibrium with trivial sunspots, they can also be associated with equilibria in which the allocation is inefficient and where drops in asset prices may even coincide with the failure of a significant number of banks. In those equilibria with non-trivial sunspots, overinvestment in production occurs if the banking sector is safe and underinvestment if the banking sector is risky. We provide an example where consumers' expected utility is higher with a safe banking sector than with a risky banking sector. Hence, our model supports the notion that financial stability is associated with higher welfare and more productive investment. As these differences are solely due to differences in the structure of the equilibrium banking sector, not due to different underlying fundamentals or sunspot probabilities, they can be directly attributed to financial stability.

Against this background, our findings have interesting implications for bank regulators. Provided there are multiple equilibria, they differ in terms of expected utilities for consumers and with respect to banks' portfolio choices. Therefore, a new potential role for liquidity regulation arises as a mechanism to select the most desirable equilibrium by requiring banks to hold a certain stock of liquid reserves. We consider different liquidity ratios and find that designing an appropriate policy can be difficult. Either those liquidity measures are non-informative or they have a non-monotonic relationship with welfare. Moreover, if liquidity requirements are set too high, the economy may be forced into an equilibrium without run-proof banks. The liquidity requirement, which is meant to promote the resilience of the banking sector, may then actually lead to greater financial fragility.

The papers closest to ours are Allen and Gale (2004a,b), Cooper and Ross (1998, 2002) and Ennis and Keister (2006). Allen and Gale (2004a,b) analyze economies with interbank asset markets.[1] There are risks to fundamentals but no coordination failures, and productive investments cannot be physically liquidated. Shocks to fundamentals have disproportionately large effects on banks and asset prices in that there are either bank failures, asset price volatility or both, causing allocative ineffciencies. If fundamentals become asymptotically deterministic, however, the equilibrium uniquely converges to one with trivial sunspots. Although asset prices are indeterminate in those equilibria, there is no real indeterminacy as the efficient allocation obtains. In our paper, we look at economies that can be regarded as the limit economy in Allen and Gale (2004a), augmented by the possibility of sunspots to trigger coordination failures. We show that equilibria with trivial sunspots still exist in these economies but the set of asset prices supporting such equilibria

[1] Starting with Allen and Gale (2000), others consider interbank deposits. For example, Skeie (2008) studies nominal contracts and Freixas et al. (2011) explore the role of monetary policies in absence of coordination failures.

is more limited. Moreover, other equilibria exist in which banks do not provide optimal liquidity insurance. In these equilibria, either (some) banks fail occasionally or there is real indeterminacy. Also in contrast to Allen and Gale (2004a), we analyze the conditions under which different types of equilibria exist and show that the endogeneity of asset values can be a source of multiplicity of equilibria.[2]

Cooper and Ross (1998, 2002) and Ennis and Keister (2006) allow for coordination failures but the value of productive investments is exogenous and does not depend on whether bank runs occur. There is a unique threshold such that a bank is run-proof if and only if the sunspot probability is above this threshold; otherwise a bank is run-prone. In equilibria where banks are run-proof, banks hold more reserves and make less productive investments than banks in equilibria where they are run-prone. In our paper, introducing a secondary interbank asset market implies a richer set of equilibrium outcomes, including multiple equilibria. In equilibria with trivial sunspots, all banks provide the first-best liquidity insurance, which cannot occur in Cooper and Ross (1998, 2002) and Ennis and Keister (2006). Also in contrast to those papers, the banking sector holds fewer reserves and makes more productive investments if all banks are run-proof, not if all are run-prone.

Starting with Jacklin (1987), a literature has developed that studies the relationship between banking mechanisms and opportunities for consumers to trade directly on markets. In Jacklin and Bhattacharya (1988) consumers can trade on equity markets, in Farhi et al. (2009) they borrow from and lend to each other, and in Diamond (1997) some consumers can trade productive assets with banks. In this literature, trading opportunities are considered to have the potential to adversely affect, or be superior to, allocations implementable by banks. We show that the allocation in any equilibrium with interbank markets for productive investments is better for consumers than if there is no market at all, and even the first-best allocation can be achieved. This suggests that it is not so much the mere existence of markets which harms efficiency of banking mechanisms but rather who is trading there. To make this point, we turn off other trading opportunities by building on two frictions. First, only banks possess the specific skills necessary to collect the returns on productive

[2] Matsuoka (2013) suggests that asymmetric equilibria may exist in environments like ours. We provide a comprehensive characterization of different equilibria including the possibility of multiple equilibria and indeterminacy.

investments (as suggested by Diamond and Rajan, 2001). Lacking such skills, consumers are not willing to buy productive investments. Second, consumers cannot commit to repay loans. Since they live for either two or three dates, penalties like future exclusion from credit markets (as in Kehoe and Levine, 1993) are ineffective for enforcing loan repayments.

Understanding bank runs as coordination failures has greatly benefited from taking an optimal contracts approach. A major conclusion from this line of research is that whether optimal contracts protect a bank from coordination failures depends on the specific combination of technologies and frictions that characterize the environment in which the bank operates (see the survey by Ennis and Keister, 2010b). Seminal papers in this field include Wallace (1988), Green and Lin (2003), Peck and Shell (2003), and Ennis and Keister (2009, 2010a). There, optimal contracts have been specified under two assumptions. First, the values of productive investments at every date are exogenous and independent from the occurrence of a sunspot. Second, it is possible to write contracts which allow payments to a consumer to be conditional on any new piece of information in the order its arrival, in particular on how many other consumers have been already served before.

We consider the values of productive investments as equilibrium outcomes. With regards to contracts, in our paper it is a bank's portfolio choice that determines whether it is immune to liquidity shocks, while the contract it offers to consumers is restricted to a simple deposit contract. With a simple contract, the bank is either able to make the fixed payment that has been promised or a prespecified rationing mechanism applies.[3] We follow this approach partly because bank regulation primarily considers a bank's portfolio structure, not so much the specific design of contracts, as decisive for how banks perform their functions. Moreover, simple contractual arrangements are justified in environments where contracts are incomplete ex ante and renegotiating them ex post is costly. Incompleteness arises as consumers often do not obtain (almost) perfect and verifiable information about the state of the world at (almost) no cost. Renegotiating deposit contracts is

[3]Cooper and Ross (1998, 2002), Ennis and Keister (2006), and particularly Allen and Gale (2004a,b) and Allen et al. (2018) also restrict attention to such simple contracts. Cooper and Ross (1998, 2002) and Ennis and Keister (2006) assume random rationing, while Allen and Gale (2004a,b) and Allen et al. (2018) consider equal rationing. Provided consumers have to agree on a rationing mechanism ex ante, equal rationing is efficient as it maximizes expected utility conditional on a bank run.

costly as it takes time, which is exactly what consumers with urgent liquidity needs often do not have. Therefore, payments are not made contingent on certain variables, even if they are observable (Rajan, 1998).[4]

In section 2 we lay out the model. In section 3 we study the properties of equilibria. In section 4 we discuss implications of our findings. Section 5 concludes.

2 The model

2.1 Setup

There are three dates $t \in \{0,1,2\}$ with a single good at every date and extrinsic risk at date $t = 1$. At this date there are two possible states $s \in \{1,2\}$. With probability $p \in]0,1[$ the state is $s = 1$ and with probability $1-p$ the state is $s = 2$.

There are two constant-returns-to-scale technologies, storage and production. Storage of the good is a short asset, also referred to as reserves. It can be used at dates $t \in \{0,1\}$ and yields a gross return of one per unit at the next date $t+1$. Production of the good is a long asset, also called productive investment. It has to be initiated at date $t = 0$ and can be physically liquidated for some arbitrarily small gross return $\varepsilon > 0$ at the interim date $t = 1$. Provided it is not liquidated, it yields a gross return of $R > 1$ per unit at the final date $t = 2$.

There is a continuum of identical consumers with mass one. A consumer has direct access to storage, but does not have the skills to initiate productive investments or to collect their returns. She is described by her endowment $(1,0,0)$ and her consumption set $X = \mathbb{R}^2_+$. A consumer is either impatient and values consumption at date $t = 1$ or patient and values consumption at date $t = 2$. At date $t = 1$ consumers learn their type, which is private information. Patience among consumers is uncorrelated and the share of impatient consumers $\lambda \in]0,1[$ is deterministic and common knowledge. Let $x_{t,s}$ denote what a consumer gets at date t in state s. Then, her expected

[4]Experiments show that simple deposit contracts make banks indeed susceptible to coordination failures (Garratt and Keister, 2009; Arifovic et al., 2013; Arifovic and Jiang, 2014; Chakravarty et al., 2014).

utility is

$$\lambda\left(pu(x_{1,1})+(1-p)u(x_{1,2})\right)+(1-\lambda)\left(pu(x_{2,1})+(1-p)u(x_{2,2})\right). \tag{1}$$

The Bernoulli utility function u is twice differentiable with $u' > 0$, $u'' < 0$, and $\lim_{x\to 0} u'(x) = \infty$. Like in many varieties of the Diamond and Dybvig (1983) model, relative risk aversion $k(x) = -xu''(x)/u'(x)$ is supposed to be larger than one. Consumers cannot commit to repay loans such that there is no credit market on which consumers can borrow from or lend to each other.

There is a continuum of identical banks with unit mass. A bank has access to storage at dates $t \in \{0,1\}$, and possesses the skills to initiate productive investments at date $t = 0$ and to collect their returns at date $t = 2$. Banks can also access a perfectly competitive interbank market for productive investments at date $t = 1$. The asset price on that market in state s is P_s. A bank offers simple deposit contracts in exchange for consumer endowments at date $t = 0$. Such contracts specify the amount a consumer is entitled to withdraw. It is not possible to write complete contracts, i.e. complex conditional payment schedules are excluded, and renegotiating contracts is prohibitively costly. Deposit contracts are therefore bound to have a simple structure. If a consumer withdraws at date $t = 1$, her claim on the bank is d, and if she withdraws at date $t = 2$, her claim is D. Following Allen and Gale (2004a,b) and Allen et al. (2018), we assume that a prespecified rationing mechanism applies once a bank is not able to meet all withdrawal demands. Given that consumers are risk averse, equal rationing maximizes expected utility. Accordingly, whenever the total claims of consumers who want to withdraw exceed the value of the bank's assets at that date, the bank splits the total asset value pro-rata among consumers and the bank ceases to exist. Without loss of generality, D can thus be set to infinity.

The market for deposits is perfectly competitive. A consumer chooses in which bank to deposit her endowment, but she has to put all her endowments in the same bank. A bank attracts a representative subset of consumers with a share of impatient consumers equal to λ, stores a share $y \in [0,1]$ of its deposits and invests a share $1-y$ in production. There is no asymmetric information about how the bank allocates deposits at date $t = 0$.

Impatient consumers always withdraw at date $t = 1$. Patient consumers can leave their deposits in the bank, but can also pretend to be impatient and withdraw. If state $s = 1$ materializes, a patient consumer compares what she gets by withdrawing at date $t = 1$ with the payoff associated with holding on until date $t = 2$, assuming *all other patient consumers withdraw early* at date $t = 1$. If the former is higher, everyone withdraws at $t = 1$. If state $s = 2$ materializes, there is no such coordination failure, yet there can be a bank failure. If a patient consumer expects that, even *without other patient consumers withdrawing early*, the value of bank assets at date $t = 2$ will not allow the bank to pay at date $t = 2$ at least as much as the promised payment to impatient consumers, she is better off by pretending to be impatient and withdraw early. The incomplete deposit contract is, therefore, not generally incentive compatible in the sense that patient consumers may find it optimal to withdraw at date $t = 1$ and not to wait until date $t = 2$.

As standard, first-best consumption for patient and impatient consumers is $R(1-y^*)/(1-\lambda) < R$ and $y^*/\lambda > 1$, respectively, and optimum storage y^* satisfies

$$u'(y^*/\lambda) = Ru'\left(\tfrac{R(1-y^*)}{1-\lambda}\right). \tag{2}$$

2.2 Bank behavior

Let $x = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$ denote the bundle of consumption $x_{t,s}$ at date t in state s. Moreover, let $N(P_s) = \max\{P_s, \varepsilon\}$ be the value of a unit of the long asset at date $t = 1$ in state s, and $M(P_s) = \max\{R/P_s, 1\}$ be the rate of return on a bank's assets between dates $t = 1$ and $t = 2$ in state s.

Banks can either take their chances, or they make provisions to prevent a possible bank run. Accordingly, banks are either run-prone or run-proof. Given perfect competition for deposits, a bank's objective is to maximize expected utility (1) subject to its constraints. These constraints are different for run-proof and run-prone banks. For a bank to be run-proof, the value of its assets at date $t = 1$ must at least cover all outstanding deposits in state $s = 1$. It is not necessary that the reserves of a run-proof bank cover all outstanding deposits. As long as depositors expect that by selling or liquidating its assets, a bank will always be able to satisfy everyone's withdrawal demand at once and in full, patient consumers do not have an incentive to run. In state $s = 2$, impatient consumers withdraw d. Patient consumers are willing to wait only if they expect to get at least d at date $t = 2$, for otherwise they would be better off withdrawing from the bank already at date $t = 1$. For the bank, which realizes a return $M(P_s)$ on its asset between dates $t = 1$ and $t = 2$, the present value of paying all patient consumers d at date $t = 2$ is $(1-\lambda)M(P_s)^{-1}d$. Therefore, for a bank to be run-proof, the value of its assets at date $t = 1$ needs to satisfy

$$\begin{aligned} d &\leq y + N(P_1)(1-y), \\ \lambda d + (1-\lambda)M(P_2)^{-1}d &\leq y + N(P_2)(1-y). \end{aligned} \tag{3}$$

The resource constraints on consumption with a run-proof bank are

$$\begin{aligned} x_{1,s} &\leq d, \\ x_{2,s} &\leq M(P_s)\frac{y + N(P_s)(1-y) - \lambda d}{1-\lambda}. \end{aligned} \tag{4}$$

The first line reflects that a run-proof bank always repays its deposits at date $t = 1$. The second requires that consumption of patient consumers is at most the pro-rata share of the future value of the bank's assets net of its liabilities to impatient consumers. Provided the asset price in state $s = 1$ satisfies $P_1 \leq 1$, a coefficient of relative risk aversion larger one has two implications. First, as the first-best consumption for impatient consumers y^*/λ is larger one, it cannot be offered by a run-proof bank. Second, a run-proof bank does not hold more reserves than needed to deter consumers from running. Consumers are simply too risk averse to be interested in speculating on fire-sales, as this would only benefit patient consumers at the expense of impatient consumers.[5]

As for a run-prone bank, there is a run caused by coordination failures in state $s = 1$ if the value of the bank's assets is not sufficient to fully pay all depositors the promised amount. There is a bank failure in state $s = 2$ unrelated to coordination failures if bank assets do not generate a

[5]See Appendix A.

sufficient return during the second period. A bank is thus run-prone if either

$$\begin{aligned} d &> y+N(P_1)(1-y), \\ \lambda d+(1-\lambda)M(P_2)^{-1}d &\leq y+N(P_2)(1-y). \end{aligned} \tag{5}$$

or

$$\begin{aligned} d &\leq y+N(P_1)(1-y), \\ \lambda d+(1-\lambda)M(P_2)^{-1}d &> y+N(P_2)(1-y). \end{aligned} \tag{6}$$

Provided a bank is prone to failure, it can fail only in one state, either in state $s=1$ or in state $s=2$. If a bank would fail in state $s=1$ as well as in state $s=2$, the marginal rate of substitution between early and late consumption would be one, regardless in which state the economy is. Since the ex-ante marginal rate of transformation is R^{-1}, this cannot be optimal.

Let θ denote the state in which a run on a run-prone bank occurs. If $\theta=1$ the run is due to a coordination failure, if $\theta=2$ it is caused by asset returns being too low. In state $s=\theta$, everyone gets a pro-rata share of the value of a bank's assets. In state $s\neq\theta$, impatient consumers get what the deposit contract entitles them to and patient consumers equally share the future value of the bank's assets net of its liabilities to impatient consumers. The budget constraints are thus

$$\begin{aligned} x_{1,s} &\leq \begin{cases} y+N(P_s)(1-y) & \text{if} \quad s=\theta, \\ d & \text{if} \quad s\neq\theta, \end{cases} \\ x_{2,s} &\leq \begin{cases} y+N(P_s)(1-y) & \text{if} \quad s=\theta, \\ M(P_s)\dfrac{y+N(P_s)(1-y)-\lambda d}{1-\lambda} & \text{if} \quad s\neq\theta. \end{cases} \end{aligned} \tag{7}$$

2.3 Interbank asset markets

Asset prices are such that arbitrage opportunities do not exist. At date $t=0$ banks have access to two assets with identical costs: the productive investment with values (P_1,P_2) and reserves with values $(1,1)$, both at date $t=1$. If $P_1,P_2\geq 1$ with $P_1+P_2>2$, arbitrage opportunities for banks would exist. By initiating production at date $t=0$ and selling the productive investment at date $t=1$, banks could realize a profit in at least one state without making a loss in the other state. However, as all banks would then invest only in production at date $t=0$, there would be no stored goods at date $t=1$ and thus productive investments cannot be sold for these prices. Similarly, if $P_1, P_2 \leq 1$ with $P_1 + P_2 < 2$, arbitrage opportunities for banks would also exist. Holding reserves at date $t=0$ and buying productive investments at date $t=1$, banks could again realize a profit in at least one state without making a loss in the other state. However, as all banks would hold only reserves and none would invest in production at all at date $t=0$, there would be no productive investments to buy for these prices. Therefore, $P_1 < 1 < P_2$, $P_2 < 1 < P_1$ or $P_1 = P_2 = 1$. Moreover, if $P_s < \varepsilon$, there would be arbitrage opportunities in that all banks could buy productive investments in state s at date $t=1$ only to liquidate them. If $P_s > R$ all banks would sell productive investments in state s at date $t=1$. As there would be no bank buying them, banks could not sell at this price and to divest productive assets they would have to be physically liquidated. Neither can be in equilibrium. Therefore, prices additionally satisfy $P_1, P_2 \geq \varepsilon$ and $P_1, P_2 \leq R$.

Let superscript $\mathscr{R}$ denote the solution to a run-prone bank's problem and superscript $\mathscr{S}$ the solution to a run-proof bank's problem. Abusing terminology slightly, liquidity demand q^D of run-prone banks of unit size (supply of investments) and liquidity supply q^S of run-proof banks of unit size (demand for investments) are

$$q^D_{s=\theta} \in \begin{cases} [-P_{s=\theta}(1-y^{\mathscr{R}}), P_{s=\theta}(1-y^{\mathscr{R}})] & \text{for } P_{s=\theta} = \varepsilon, \\ \{P_{s=\theta}(1-y^{\mathscr{R}})\} & \text{for } P_{s=\theta} > \varepsilon, \end{cases} \tag{8a}$$

$$q^D_{s\neq\theta} \in \begin{cases} \{\lambda d^{\mathscr{R}} - y^{\mathscr{R}}\} & \text{for } P_{s\neq\theta} < R, \\ [\lambda d^{\mathscr{R}} - y^{\mathscr{R}}, P_{s\neq\theta}(1-y^{\mathscr{R}})] & \text{for } P_{s\neq\theta} = R, \end{cases} \tag{8b}$$

and

$$q_s^S \in \begin{cases} \left[y^{\mathscr{S}} - \lambda d^{\mathscr{S}}, y^{\mathscr{S}} + P_s(1-y^{\mathscr{S}}) - \lambda d^{\mathscr{S}}\right] & \text{for } P_s = \varepsilon, \\ \{y^{\mathscr{S}} - \lambda d^{\mathscr{S}}\} & \text{for } \varepsilon < P_s < R, \\ \left[-P_s(1-y^{\mathscr{S}}), y^{\mathscr{S}} - \lambda d^{\mathscr{S}}\right] & \text{for } P_s = R. \end{cases} \tag{9}$$

In state $s = \theta$, bank runs occur and run-prone banks sell all their assets $(1 - y^{\mathscr{R}})$ if the asset price is larger than the liquidation value, else they are indifferent between selling and liquidating. In state $s \neq \theta$ they possess reserves of $y^{\mathscr{R}}$ and pay $\lambda d^{\mathscr{R}}$ to impatient consumers. Hence, they sell assets if doing so is necessary to pay the promised amounts to their impatient consumers. Provided storage exceeds promised payments, they either buy assets if $P_{s\neq\theta} < R$ or are indifferent between holding, buying or selling productive assets if $P_{s\neq\theta} = R$. Regarding run-proof banks, since patient consumers have no incentive to ever withdraw early, the actual outflow in both states is $\lambda d^{\mathscr{S}}$. Moreover, since the bank's decision about $y^{\mathscr{S}}$ and $d^{\mathscr{S}}$ is made at date $t = 0$, i.e. before the extrinsic risk is resolved, net reserves at date $t = 1$, $y^{\mathscr{S}} - \lambda d^{\mathscr{S}}$, are state-independent if prices in both states satisfy $\varepsilon < P_s < R$. In principle, this amount can be positive or negative. For $P_s = R$ run-proof banks are indifferent between buying and selling and for $P_s = \varepsilon$ they are indifferent between holding on to their own productive assets and liquidating them for the purpose of buying productive assets from run-prone banks.

Let ρ be the share of consumers who put their endowments in run-prone banks, or the share of run-prone banks for short. Then, Q_s^D and Q_s^S denote aggregate liquidity demand and aggregate liquidity supply, respectively, with

$$\begin{aligned} Q_s^D &= \rho q_s^D, \\ Q_s^S &= (1-\rho) q_s^S. \end{aligned} \tag{10}$$

3 Equilibrium banking sectors

3.1 Equilibrium concept and existence

It is convenient to simplify some notation. A consumption plan (x^τ, d^τ, y^τ) for a consumer who deposits her endowments with a bank of type $\tau \in \{\mathscr{S}, \mathscr{R}\}$ is a consumption bundle x^τ and a bank portfolio (d^τ, y^τ) satisfying the constraints (3) and (4) for $\tau = \mathscr{S}$, and either (5) or (6) together with (7) for $\tau = \mathscr{R}$. Moreover, for given prices $\mathbf{P} = (P_1, P_2)$, let $V^\tau(\mathbf{P})$ denote the indirect utility offered to consumers by a bank of type τ.

Definition 1 *For a given probability distribution of the extrinsic state, an* ***equilibrium*** *is a set of consumption plans, asset prices and the share of run-prone banks*

$$\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), \mathbf{P}, \rho\right)$$

with the following properties:

- *Banks maximize expected utility:* $(y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}})$ *is a solution to the consumer problem for run-proof banks, and* $(y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}})$ *is a solution to the consumer problem for run-prone banks.*

- *The interbank market clears:*

$$Q_s^D = Q_s^S \quad \textit{for} \quad s = 1, 2.$$

- *Consumers are not better off by going to another operating bank:*

$$\begin{aligned} V^{\mathscr{S}}(\mathbf{P}) &= V^{\mathscr{R}}(\mathbf{P}) \quad &\textit{if} \quad \rho \in]0,1[, \\ V^{\mathscr{S}}(\mathbf{P}) &\geq V^{\mathscr{R}}(\mathbf{P}) \quad &\textit{if} \quad \rho = 0, \\ V^{\mathscr{S}}(\mathbf{P}) &\leq V^{\mathscr{R}}(\mathbf{P}) \quad &\textit{if} \quad \rho = 1. \end{aligned}$$

Our first result is that equilibria exist.

Theorem 1 *There is an equilibrium for every probability distribution.*

Proof: See Appendix B.1 □

An equilibrium always exists, although solving for it is difficult. However, key insights arise from the solutions to the banks' problems. No-arbitrage implies that prices are such that $N(P_s) = P_s$ and $M(P_s) = R/P_s$. Non-satiation implies that the budget constraints (4) and (7) hold with equality. For a run-proof bank, for which the first line in condition (3) is binding, replacing d by $y + P_1(1-y)$ allows to express the objective function solely in terms of y. As the problem is convex, its solution is unique and, if interior, solves the first-order condition

$$\begin{aligned}\left(\tfrac{1}{R}\tfrac{\lambda}{1-\lambda}u'\left(y+P_1(1-y)\right)+\tfrac{p}{P_1}u'\left(\left(y+P_1(1-y)\right)\tfrac{R}{P_1}\right)\right)(1-P_1)\\ -\tfrac{1-p}{P_2}u'\left(\tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{1-\lambda}\right)\left(P_2-1+\tfrac{\lambda}{1-\lambda}(P_2-P_1)\right)=0.\end{aligned} \tag{11}$$

As for a run-prone bank, we replace $x_{t,s}$ accordingly in the objective function, which is then expressed in terms of y and d. Again, the problem is convex and the solution $(d^{\mathscr{R}}, y^{\mathscr{R}})$ is thus unique. The probability of the state in which the bank fails is $\Pr(s=\theta) = p$ if it fails in state $s = 1$ and it is $\Pr(s = \theta) = 1 - p$ if the bank fails in state $s = 2$. The first-order conditions then read

$$\frac{u'(d)}{u'\left(\frac{R}{P_{s\neq\theta}}\frac{y+P_{s\neq\theta}(1-y)-\lambda d}{1-\lambda}\right)}-\frac{R}{P_{s\neq\theta}} \;=\; 0, \tag{12a}$$

$$\frac{u'\left(y+P_{s=\theta}(1-y)\right)}{u'\left(\frac{R}{P_{s\neq\theta}}\frac{y+P_{s\neq\theta}(1-y)-\lambda d}{1-\lambda}\right)}-\frac{\Pr(s\neq\theta)}{\Pr(s=\theta)}\frac{P_{s\neq\theta}-1}{1-P_{s=\theta}}\frac{R}{P_{s\neq\theta}} \;\leq\; 0, \tag{12b}$$

with strict inequality in the second line if $y^{\mathscr{R}} = 0$. Finally, the solution to the unconstrained optimization problem, that is ignoring conditions (3), (5) and (6), satisfies the following first-order conditions

$$u'(d) = R\left(u'\left(\tfrac{R}{P_1}\tfrac{y+P_1(1-y)-\lambda d}{(1-\lambda)}\right)\frac{p}{P_1} + u'\left(\tfrac{R}{P_2}\tfrac{y+P_2(1-y)-\lambda d}{(1-\lambda)}\right)\frac{1-p}{P_2}\right), \tag{13a}$$

$$u'\left(\tfrac{R}{P_1}\tfrac{y+P_1(1-y)-\lambda d}{(1-\lambda)}\right) = -\frac{1-p}{p}\frac{P_1}{1-P_1}\frac{1-P_2}{P_2}u'\left(\tfrac{R}{P_2}\tfrac{y+P_2(1-y)-\lambda d}{(1-\lambda)}\right). \tag{13b}$$

3.2 Equilibria with trivial sunspots

In accordance with Allen and Gale (2004a), equilibria with trivial sunspots are defined as follows.

Definition 2 *Suppose* $\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), \mathbf{P}, \rho\right)$ *is an equilibrium. It is an equilibrium with **trivial sunspots** if asset prices differ across extrinsic states and the first-best allocation obtains.*

The first-best allocation requires that the consumption of patient and impatient consumers does not depend on the extrinsic state. Suppose banks can make an unconstrained choice. According to the first-order condition (13b), consumption of patient consumers is state-independent provided prices satisfy $p/P_1 + (1-p)/P_2 = 1$, or equivalently $P_2 = (1-p)/(1-p/P_1)$, which has two immediate effects. First, one unit invested in storage at date $t = 0$ and used to buy productive investments at date $t = 1$ has the same expected return at date $t = 2$ as one unit invested in production at date $t = 0$. Second, the banks' liquidity supply is zero as for those prices condition (13b) only holds if $\lambda d^{\mathscr{S}} = y^{\mathscr{S}}$. These effects together imply that for $p/P_1 + (1-p)/P_2 = 1$, condition (13a) is equivalent to $u'\left(d^{\mathscr{S}}\right) = Ru'\left(R(1-\lambda d^{\mathscr{S}})/(1-\lambda)\right)$, i.e. the first-best allocation $d^{\mathscr{S}} = y^*/\lambda$ obtains.

State-independent consumption for impatient consumers can only be provided by run-proof banks. Consumption offered through run-prone banks necessarily depends on the extrinsic state as there will be a run in exactly one of the extrinsic states. According to condition (3), however, run-proof banks can implement the efficient allocation only if $y^*/\lambda \leq y^* + P_1(1-y^*)$, or equivalently if $P_1 \geq (1/\lambda - 1)(1/y^* - 1)^{-1}$. We conclude:

Theorem 2 *Let* $p^T := (1-\varepsilon)/(1-\varepsilon\frac{\lambda}{1-\lambda}\frac{1-y^*}{y^*})$. *Equilibria with trivial sunspots exist if and only if* $p \leq p^T$. *In such equilibrium asset prices are indeterminate and satisfy*

$$
\begin{aligned}
P_1 &\in \left[(1/\lambda-1)(1/y^*-1)^{-1}, R\right], \\
P_2 &= (1-p)/(1-p/P_1).
\end{aligned}
$$

Proof: See Appendix B.2. □

Several interesting implications arise. First, the mere possibility of coordination failures does not necessarily entail bank runs or that banks cannot provide efficient liquidity insurance. Second, because run-prone banks cannot make consumers better off in a bank run than by splitting asset values equally, those banks cannot offer contracts that give consumers higher expected utility and hence do not exist in equilibria with asset prices as in Theorem 2.[6] Third, since p^T is below but arbitrarily close to one as ε is arbitrarily close to zero, there can be a wide range of probability distributions for which equilibria with trivial sunspots exist. Finally, while such equilibria also exist in economies where coordination failures are ruled out (Allen and Gale, 2004a), in our economies where coordination failures are possible asset prices not only have to satisfy the first condition $P_2 = (1-p)/(1-p/P_1)$ but additionally $P_1 \geq (1/y^*-1)^{-1}(1/\lambda-1)$. As relative risk aversion is greater one, we have $y^* > \lambda$. Therefore, neither equilibria with stable asset prices $P_1 = P_2 = 1$ nor with state-dependent asset prices satisfying $P_1 < 1$ support the efficient allocation.

That equilibria with trivial sunspots are inconsistent with $P_1 \leq 1$ means that with trivial sunspots, consumers contemplate to run in the extrinsic state in which the asset price is strictly larger than in the other state. One would expect, however, that consumers consider to run particularly when the value of bank assets is low. This bears the question whether other types of equilibria exist, of which there are potentially three.

[6] Also, deviating from our setup by imposing a sequential service constraint, Theorem 2 implies that such a constraint is not necessarily binding in equilibrium.

Definition 3 *Suppose* $\big((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), \mathbf{P}, \rho\big)$ *is an equilibrium in which* $P_1 \leq P_2$ *obtains. It is an equilibrium with a* ***safe banking sector*** *if* $\rho = 0$*; with a* ***risky banking sector*** *if* $\rho = 1$*; and with a* ***mixed banking sector*** *if* $\rho \in]0,1[$.

3.3 Safe banking sectors

We begin with equilibria with a safe banking sector and stable asset prices.

Theorem 3 *There is a* $\check{p} < 1$ *such that an equilibrium with a safe banking sector and stable asset prices exists if and only if* $p \geq \check{p}$*. In such equilibrium*

- *banks' reserves satisfy* $y^{\mathscr{S}} = \lambda$*;*
- *consumers' expected utility is strictly lower than the first-best expected utility.*

Proof: See Appendix B.3 □

Arbitrage-free asset prices are equal across states only if $P_1 = P_2 = 1$. As structuring its portfolio at $t = 0$ is then as good for any bank as structuring it at $t = 1$, an individual bank's reserves are indeterminate. If at date $t = 1$, an individual bank's reserves are less than required to pay impatient consumers a total of $\lambda d^{\mathscr{S}}$ the bank will sell productive investment, and if an individual bank holds reserves above $\lambda d^{\mathscr{S}}$ it will buy productive investment. In aggregate, however, all run-proof banks together hold just sufficient reserves to pay out all depositors at $t = 1$, i.e. $\lambda d^{\mathscr{S}} = y^{\mathscr{S}}$. Note that trade of assets at $t = 1$ does not affect the consumption for impatient or patient consumers. For $P_1 = P_2 = 1$, run-proof banks pay one unit of the good to impatient consumers, and R units to patient consumers. Feasibility thus requires that banks' aggregate reserve holdings are equal to the share λ of impatient consumers.

Safe banking sectors may not only exist for $\mathbf{P} = (1,1)$. In any equilibrium without run-prone banks there is no liquidity demand from those banks. Hence, $q^S = 0$ must hold for $\rho = 0$. According to equation (9), provided asset prices are bounded away from the return of production R as well as from its liquidation value ε, a necessary and sufficient condition for $q^S = 0$ is $\lambda d^{\mathscr{S}} = y^{\mathscr{S}}$. As banks are run-proof provided $d^{\mathscr{S}} = y^{\mathscr{S}} + P_1(1 - y^{\mathscr{S}})$, the liquidity supply by run-proof banks is therefore zero if $y^{\mathscr{S}} = \lambda P_1 / (\lambda P_1 + 1 - \lambda)$ and $d^{\mathscr{S}} = P_1 / (\lambda P_1 + 1 - \lambda)$. Let h be a correspondence such that for $P_1 \in [\varepsilon, 1]$

$$h(P_1) = \left\{ P_2 \in [1, R] \,\middle|\, P_2 \text{ satisfy (11) and } y^{\mathscr{S}} = \lambda P_1 / (\lambda P_1 + 1 - \lambda) \right\}. \tag{14}$$

Then, the solution to the optimization problem of run-proof banks implies a that their liquidity supply is zero provided asset prices satisfy $P_2 = h(P_1)$. If $h(P_1) = \emptyset$ then P_1 is incompatible with a zero-liquidity supply. For $h(P_1) \neq \emptyset$, the correspondence h satisfies

$$h(P_1) = \frac{\lambda P_1 + (1 - \lambda)}{1 - \frac{1}{1-p} \frac{1-P_1}{P_1} \left(\lambda \frac{u'\left(\frac{P_1}{\lambda P_1 + 1 - \lambda}\right)}{u'\left(\frac{R}{\lambda P_1 + 1 - \lambda}\right)} \frac{P_1}{R} + p(1 - \lambda) \right)}, \tag{15}$$

that is, it becomes a continuous and monotonically decreasing function for $P_1 \in [h^{-1}(R), 1]$, with $h(1) = 1$, $h^{-1}(R) > \varepsilon$ and $\lim_{p \to 1} h^{-1}(R) = 1$. Liquidity supply from the group of run-proof banks is positive for all $P_1 < h^{-1}(P_2)$ and negative for all $P_1 > h^{-1}(P_2)$.[7]

Theorem 4 *Suppose* $\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), \mathbf{P}, \rho\right)$ *is an equilibrium with a safe banking sector and stable asset prices. Provided* $V^{\mathscr{S}}(\mathbf{P}) > V^{\mathscr{R}}(\mathbf{P})$ *for* $\mathbf{P} = (1,1)$, *there are other equilibria with a safe banking sector and* $P_1 < 1$. *In such equilibrium*

- *asset prices and consumption are indeterminate;*
- *banks' reserves satisfy* $y^{\mathscr{S}} < \lambda$*;*
- *banks' reserves are the lower the lower the asset price* P_1 *is.*

Proof: See Appendix B.4 □

[7] This is because the first-order condition (11) implicitly defines $y^{\mathscr{S}}$ as a function of P_2 for any given P_1. Evaluated at $y^{\mathscr{S}} = \lambda P_1 / (\lambda P_1 + 1 - \lambda)$, this function satisfies $dy^{\mathscr{S}} / dP_2 < 0$. For every $P_1 \in [h^{-1}(R), 1]$ there is a unique P_2 such that $q^S = 0$. Therefore, $y^{\mathscr{S}} > \lambda P_1 / (\lambda P_1 + 1 - \lambda)$ and thus $q^S > 0$ for all $P_1 < h^{-1}(P_2)$ (and vice versa).

A sufficient condition for $V^{\mathscr{S}}(1,1) > V^{\mathscr{R}}(1,1)$ is $p > \check{p}$. According to the Theorem, a continuum of prices, bank balance sheets and consumption allocations then exists that is supported by a safe banking sector. Asset prices are indeterminate because if run-proof banks offer a strictly better expected utility than run-prone banks for $\mathbf{P} = (1,1)$, asset prices can deviate somewhat from $\mathbf{P} = (1,1)$ and run-proof banks are still the better choice. This also applies to any combination of asset prices in some neighborhood of $\mathbf{P} = (1,1)$ that satisfy the zero-liquidity supply condition (14). Note, there is no trade with run-prone banks at date $t = 1$ because there are none, and trade among run-proof banks does not affect the total value of resources available to each one of them. Therefore, with $(y^{\mathscr{S}}, d^{\mathscr{S}})$ being set at date $t = 0$, consumption does not depend on the extrinsic state. Consumption depends, however, on asset prices and is thus also indeterminate. Impatient consumers get $P_1/(\lambda P_1 + 1 - \lambda)$ and patient consumers get $R/(\lambda P_1 + 1 - \lambda)$.

3.4 Risky banking sectors

Without run-proof banks, there is no supply of reserves upon which run-prone banks could rely at the interim date, regardless in which state they are. Hence, for banking sectors to be risky, liquidity demand by the group of run-prone banks is necessarily zero in both states. In state $s = 1$, liquidity demand is zero if and only if the asset price is not larger than the physical liquidation value of assets: banks weakly prefer to liquidate production over selling. In state $s = 2$, liquidity demand is zero if and only if the asset price is such that reserves held by a run-prone bank exactly cover its total payout to impatient consumers. However, the optimal consumption plan requires that the marginal rate of substitution between consumption when patient and when impatient is equal to the rate of return on holding the long asset between date 1 and date 2; see first-order condition (12a). No-arbitrage implies that there is a lower bound for this rate of return. Hence, for given reserves, consumption of patient consumers has an upper bound. Therefore, we obtain the following result.

Lemma 1 *Suppose* $\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), \mathbf{P}, \rho\right)$ *is an equilibrium and let*

$$\hat{p} := \frac{R-1}{R-1+u'\left(\frac{\lambda R}{\lambda R+1-\lambda}\right)/u'\left(\frac{R}{\lambda R+1-\lambda}\right)}.$$

Then the banking sector cannot be risky in equilibrium if $p > \hat{p}$.

Proof: See Appendix B.5 □

The upper bound $\hat{p}$ on the sunspot probability is smaller than $(R-1)/R < 1$ and depends on the fundamentals of the economy. It is the lower the smaller the share of early consumers λ is. The effects of the return on the long asset R on $\hat{p}$ are generally not clear-cut. On the one hand, for given prices a larger R increases the rate of return on holding the long asset between date 1 and date 2. On the other hand, a larger R also changes the optimum consumption profile for consumers in case of a run compared to what they get as late consumers in case there is no run. If the coefficient of relative risk aversion is constant, $k(x) = \kappa$, we have $\hat{p} = (R-1)/(R-1+\lambda^{-\kappa})$ and the net effect is clear since $d\hat{p}/dR > 0$. Moreover, we also obtain $d\hat{p}/d\kappa < 0$.

Zero liquidity demand in both states is necessary but not sufficient for risky banking sectors to exist. Run-prone banks must also offer deposit contracts which generate a higher expected utility than deposit contracts offered by run-proof banks. This leads to our next main result.

Theorem 5 *There is a* $\bar{p} > 0$ *with* $\bar{p} \leq \hat{p}$ *such that for all* $p \leq \bar{p}$ *an equilibrium with a risky banking sector exists. In such equilibrium*

- *asset prices and consumption are determinate;*
- *banks' reserves satisfy* $y^{\mathscr{R}} > y^*$*;*
- *consumers' expected utility is strictly lower than the first-best expected utility.*

Proof: See Appendix B.6 □

In an equilibrium with a risky banking sector, all banks survive in one state and none survives in the other state. If the extrinsic state with coordination failure materializes, all banks are forced to give up their long assets. As there is no run-proof bank supplying any reserves, run-prone banks as a group have to physically liquidate all their assets. This is an equilibrium if coordination failures are sufficiently unlikely: For a run-proof bank, prospects of buying assets at fire sale prices are slim while fending off a bank run to be able to buy assets from distressed banks is costly because it requires a bank to hold large reserves relative to what it promises to impatient consumers. With a risky banking sector, the first-order conditions (12a) and (12b) read

$$0 = pu'(y^{\mathscr{R}}) + (1-p)\left(u'\left(\frac{y^{\mathscr{R}}}{\lambda}\right) - Ru'\left(\frac{R(1-y^{\mathscr{R}})}{(1-\lambda)}\right)\right), \tag{16a}$$

$$P_2 = R\frac{u'\left(\frac{R(1-y^{\mathscr{R}})}{(1-\lambda)}\right)}{u'\left(\frac{y^{\mathscr{R}}}{\lambda}\right)}. \tag{16b}$$

The first equation uniquely defines the reserves $y^{\mathscr{R}}$, and for given reserves the second equation defines a unique P_2. The consumption plan is the same as in the absence of an asset market.

3.5 Mixed banking sectors

If run-prone banks sell their assets in a bank run, no productive investment will ever go to waste. If run-proof banks can buy additional productive investments, their excess reserves are not idle but available to run-prone banks without jeopardizing the stability of run-proof banks. There are thus potentially gains from trading the extrinsic risk with each other. In an equilibrium with a mixed banking sector, such trades take place. It arises as the result of an equilibrium in mixed strategies. With probability ρ a consumer goes to a run-prone bank and with probability $1-\rho$ to a run-proof bank. Whether such an equilibrium exists depends on whether there are feasible asset prices for which liquidity supply is positive, liquidity demand is positive and state-independent, and both types of banks are equally good to consumers. State-independent liquidity demand is required because liquidity supply is state-independent and markets have to clear in all states.

According to the demand schedules (8a) and (8b), liquidity demand is state-independent if and only if $d^{\mathscr{R}} = (P_1(1-y^{\mathscr{R}}) + y^{\mathscr{R}})/\lambda$. Since $P_1 > 0$ we conclude:

Corollary 1 *A run-prone bank never holds reserves larger than the withdrawal demands in the state in which no bank run occurs.*

Moreover, for a run-prone bank, consumption by patient consumers is $x_{2,2}^{\mathscr{R}} = \frac{R}{P_2}\frac{(P_2-P_1)(1-y^{\mathscr{R}})}{1-\lambda}$ according to the budget constraint (7). Consumption is thus positive only if $P_1 < P_2$. According to conditions (5) and (6), this implies:

Corollary 2 *If bank runs occur in equilibrium, then only because of coordination failures and not because of returns on bank assets being too low.*

To derive feasible prices that induce run-prone banks to find it optimal to set $y^{\mathscr{R}}$ and $d^{\mathscr{R}}$ such that liquidity demand is state-independent, we define a correspondence f such that for $P_1 \in [\varepsilon, 1]$

$$f(P_1)=\begin{cases} \left\{(y^{\mathscr{R}},P_2) \in \{0\} \times [1,R] \;\middle|\; (y^{\mathscr{R}},d^{\mathscr{R}}) \text{ satisfy (12a) and } d^{\mathscr{R}}=P_1/\lambda \right\}, \\ \\ \left\{(y^{\mathscr{R}},P_2) \in]0,1] \times [1,R] \;\middle|\; (y^{\mathscr{R}},d^{\mathscr{R}}) \text{ satisfy (12a), (12b) and } d^{\mathscr{R}}=\frac{y^{\mathscr{R}}+P_1(1-y^{\mathscr{R}})}{\lambda} \right\}. \end{cases} \tag{17}$$

If $f(P_1) = \emptyset$, then P_1 is incompatible with state-independent liquidity demand. For $f(P_1) \neq \emptyset$, let $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2)$ denote a solution to equation (17). Then, $(y^{\mathscr{R}}, d^{\mathscr{R}})$ is a solution to a run-prone bank's optimization problem and the implied liquidity demand is state-independent provided $y^{\mathscr{R}} = \mathbf{y}^{\mathscr{R}}$ and $d^{\mathscr{R}} = \left(P_1(1-\mathbf{y}^{\mathscr{R}}) + \mathbf{y}^{\mathscr{R}}\right)/\lambda$. There are potentially many solutions for a given P_1.

As for the indifference of consumers between banks of different types, note first that according to the Envelope theorem, indirect utilities $V^{\mathscr{R}}(\mathbf{P})$ and $V^{\mathscr{S}}(\mathbf{P})$ are characterized by

$$\frac{dV^{\mathscr{R}}(\mathbf{P})}{dP_2} = (1-p)u'\left(x_{2,2}^{\mathscr{R}}\right)\frac{R}{P_2}\frac{q_2^D}{P_2} \in \begin{cases} \mathbb{R}_{++} & \text{if } q_2^D > 0, \\ \{0\} & \text{if } q_2^D = 0, \\ \mathbb{R}_{-} & \text{if } q_2^D < 0, \end{cases} \tag{18a}$$

$$\frac{dV^{\mathscr{S}}(\mathbf{P})}{dP_2} = -(1-p)u'\left(x_{2,2}^{\mathscr{S}}\right)\frac{R}{P_2}\frac{q^S}{P_2} \in \begin{cases} \mathbb{R}_{-} & \text{if } q^S > 0, \\ \{0\} & \text{if } q^S = 0, \\ \mathbb{R}_{++} & \text{if } q^S < 0, \end{cases} \tag{18b}$$

$$\frac{dV^{\mathscr{R}}(\mathbf{P})}{dP_1} = p(1-y^{\mathscr{R}})u'\left(x_{1,1}^{\mathscr{R}}\right) > 0. \tag{18c}$$

The sign of $dV^{\mathscr{S}}(\mathbf{P})/dP_1$ is not clear. Let g be a correspondence such that for $P_1 \in [\varepsilon, 1]$

$$g(P_1) = \left\{P_2 \in [1,R] \,\middle|\, q_2^D > 0, q^S > 0 \text{ and } V^{\mathscr{R}}(\mathbf{P}) - V^{\mathscr{S}}(\mathbf{P}) = 0\right\}. \tag{19}$$

If $P_2 = g(P_1)$, a consumer is indifferent between run-proof and run-prone banks. Provided $g(P_1) = \emptyset$ for a given P_1, there is no P_2 such that run-prone and run-proof banks are equally good from a consumers perspective. Either run-prone banks are strictly better than run-proof banks or run-proof banks are strictly better than run-prone banks for this P_1 regardless P_2.

Provided $g(P_1) \neq \emptyset$, the above characteristics of the indirect utilities imply that the correspondence g is an injective function and a consumer strictly prefers a run-prone bank over a run-proof bank if and only if $P_2 > g(P_1)$. A higher asset price in state $s = 2$ makes a run-prone bank more attractive because it can offer more consumption to patient consumers while holding fewer reserves. It makes a run-proof bank less attractive because its patient consumers get less as the bank cannot buy as many long assets in state $s = 2$ in exchange for a given amount of excess reserves.

Let ϕ be the projection of f, as defined in equation (17), on the P_2-coordinate. Then, a mixed banking sector is characterized by asset prices (P_1, P_2) and a share of run-prone banks ρ for which $P_1 \in]\varepsilon, 1]$, $\phi(P_1) = g(P_1) \neq \emptyset$, $P_2 = \phi(P_1)$ and

$$\rho = \frac{y^{\mathscr{S}} - \lambda d^{\mathscr{S}}}{\left(y^{\mathscr{S}} - \lambda d^{\mathscr{S}}\right) - \left(y^{\mathscr{R}} - \lambda d^{\mathscr{R}}\right)}. \tag{20}$$

Unfortunately, it is difficult to explicitly state the circumstances under which a mixed banking sector exists. However, we are able to specify two conditions that are sufficient to rule out a mixed banking sector. Recall Theorem 3 which has established $p \geq \check{p}$ as necessary and sufficient condition for an equilibrium with run-proof banking sectors and stable asset prices. Satisfying this condition does not exclude though that other equilibria with run-prone banks may also exist.

Theorem 6 *There is a $\tilde{p} \in [\check{p}, 1[$ such that for all $p > \tilde{p}$, run-prone banks cannot coexist with run-proof banks in equilibrium.*

Proof: See Appendix B.7 □

Suppose there is scope for run-prone banks to exist for some $p > \check{p}$. A sufficient condition that there is some larger probability $\tilde{p}$ above which no run-prone bank operates is that run-prone banks do not exist if the sunspot probability converges to one. To begin with, risky banking sectors do not exist then (see Lemma 1). Moreover, market clearing in both states implies that the asset price in state $s = 1$ converges to one. Hence, given the (almost) certainty of coordination failures, even if run-prone banks make productive investments, their returns are (almost) never collected and the total asset value of run-prone banks is (almost) always equal to one. Accordingly, run-prone banks do not provide any meaningful liquidity insurance and the best they can do for consumers is just about as good as storage. Run-proof banks, however, always collect the returns on the productive investments they make. They also offer at least some liquidity insurance. Hence, only run-proof banks will exist in equilibrium.

Similarly, satisfying the conditions in Lemma 1 and Theorem 5 does not rule out other equilibria in which run-proof banks exist.

Theorem 7 *There is a* $\breve{p} \in]0, \bar{p}]$ *such that for all* $p < \breve{p}$*, run-proof banks cannot coexist with run-prone banks in equilibrium.*

Proof: See Appendix B.8 □

Suppose there is scope for run-proof banks to exist for some $p \leq \bar{p}$. A sufficient condition that there is some smaller probability $\breve{p}$ below which no run-proof bank operates is that run-proof banks never exist if the sunspot probability converges to zero. Clearly, safe banking sectors cannot exist then. As for mixed banking sectors, state-independent liquidity demand by run-prone banks holding at least some reserves themselves requires that P_2 converges to one regardless which P_1 holds. Run-prone banks provide (almost) the first-best liquidity insurance. Run-proof banks do not make any productive investments and thus cannot match the expected utility offered by a run-prone bank. If run-prone banks would not hold any reserves, prices that ensure state-independent liquidity demand also imply that run-prone banks offer an expected utility higher than the first-best. Since all banks offering better contracts than in the first-best is not feasible, only run-prone banks exist in equilibrium.

To sum up, mixed banking sectors require that run-prone and run-proof banks coexist in equilibrium. Therefore, mixed banking sectors are feasible only for probability distributions of the extrinsic state for which neither run-prone banks nor run-proof banks are ruled out, i.e. for $p \in]\breve{p}, \tilde{p}[$.

3.6 Numerical examples

The following examples illustrate two features we cannot prove in general. One is that mixed banking sectors may exist, the other that multiple equilibria potentially exist of which at least two feature non-trivial sunspots. Let the Bernoulli utility function be $u(x) = -x^{-1}$, i.e. relative risk aversion is $k(x) = 2$, and the physical liquidation value be $\varepsilon = 10^{-29}$.[8] Liquidity demand is state-independent for $(y^{\mathscr{R}}, P_2) = f(P_1)$ with f as defined in equation (17). The projection ϕ of f on the

[8] We chose an arbitrary, small value. It is strictly positive to rule out an infinite return on bank assets at $t = 1$.

P_2-coordinate thus satisfies

$$\phi^{-1}(P_2) = \begin{cases} 1 - \frac{1-p}{p}\lambda^2 (P_2 - 1) & \text{if} \quad y^{\mathscr{R}} \in]0,1[, \\ P_2 \left(1 + \frac{1-\lambda}{\lambda}\left(\frac{P_2}{R}\right)^{0.5}\right)^{-1} & \text{if} \quad y^{\mathscr{R}} = 0. \end{cases} \tag{21}$$

Liquidity supply is zero for $P_2 = h(P_1)$ with h as defined in equation (14), i.e.

$$h(P_1) = \frac{1-\lambda+\lambda P_1}{1 - \frac{1}{1-p}\frac{1-P_1}{P_1}(\lambda R/P_1 + (1-\lambda)\,p)}. \tag{22}$$

The condition for indifference between bank types is $P_2 = g(P_1)$ with g as defined in equation (19). Instead of deriving g explicitly, we calculate and compare indirect utilities with run-proof and run-prone banks, respectively, for prices satisfying $P_1 = \min\left\{\phi^{-1}(P_2), h^{-1}(P_2)\right\}$. For $\min\left\{\phi^{-1}(P_2), h^{-1}(P_2)\right\} = \phi^{-1}(P_2)$, price combinations for which indirect utilities are equal constitute an equilibrium with a mixed banking sector. We then calculate d^{τ} and y^{τ} for $\tau \in \{\mathscr{R}, \mathscr{S}\}$, and the implied individual liquidity demand and supply determine the share ρ of run-prone banks according to equation (20).

Example 1 For $R = 5$, $\lambda = 0.7$ and $p = 0.17$, there are equilibria with trivial sunspots and a mixed banking sector is an equilibrium with non-trivial sunspots:

$$\rho = 0, \qquad P_1 \in \left[\sqrt{5}, 5\right] \qquad P_2 = \frac{0.83}{1-0.17/P_1}, \quad V(\mathbf{P}) = -0.696;$$

$$\rho = 0.836239, \quad P_1 = 0.306249, \quad P_2 = 1.289987, \quad V(\mathbf{P}) = -0.767.$$

Example 2 For $R = 5$, $\lambda = 0.4$ and $p = 0.13275$, there are equilibria with trivial sunspots and a safe as well as a risky banking sector are equilibria with non-trivial sunspots:

$$\rho = 0, \quad P_1 \in \left[\sqrt{5}, 5\right] \quad P_2 = \tfrac{0.86725}{1-0.13275/P_1}, \quad V(\mathbf{P}) = -0.447;$$

$$\rho = 0, \quad P_1 = 1, \qquad P_2 = 1, \qquad V(\mathbf{P}) = -0.520;$$

$$\rho = 1, \quad P_1 = \varepsilon, \qquad P_2 = 1.956688, \qquad V(\mathbf{P}) = -0.603.$$

Example 3 For $R = 5$, $\lambda = 0.4$ and $p = 0.132$, there are equilibria with trivial sunspots and a risky banking sector is an equilibrium with non-trivial sunspots:

$$\rho = 0, \quad P_1 \in \left[\sqrt{5}, 5\right] \quad P_2 = \tfrac{0.868}{1-0.132/P_1}, \quad V(\mathbf{P}) = -0.447;$$

$$\rho = 1, \quad P_1 = \varepsilon, \qquad P_2 = 1.950461, \quad V(\mathbf{P}) = -0.594.$$

Comparing example 2 with example 3 reveals that a higher sunspot probability (up from p=0.132 in example 3 to p=0.13275 in example 2) can change the set of equilibria in a way such that the maximum expected utility obtained in non-trivial sunspot equilibria is larger (up from $V(\mathbf{P}) = -0.594$ in example 3 to $V(\mathbf{P}) = -0.520$ in example 2). This is because with the higher sunspot probability a safe banking sector becomes another possible equilibrium which outperforms the risky banking sector in either of the examples.

4 Comparing equilibria

The first immediate conclusion from our analysis is that while banks could be run-proof and provide the efficient level of liquidity insurance, other equilibria potentially coexist in which the allocation is inefficient and, occasionally, (some) banks may fail when asset prices drop. In the first type of equilibrium, sunspots are trivial but asset prices are indeterminate. In equilibria with safe banking sectors and non-trivial sunspots, asset prices are also indeterminate but so is consumption.

In equilibria with at least some run-prone banks, the allocation is inefficient but asset prices and consumption are determinate.

In equilibria in which run-prone banks operate, expected utility for depositors of a run-proof bank is equal to the expected utility for depositors of a run-prone bank if the share of run-prone banks is $\rho \in]0,1[$, or smaller if the share is $\rho = 1$. To compare any two equilibria with run-prone banks, it thus suffices to look at the indirect expected utility for depositors of a run-prone bank at prices for which liquidity demand is state-independent. For $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2) = f(P_1)$, this indirect utility is

$$\begin{aligned} V^{\mathscr{R}}(\mathbf{P}) = & \, pu\left(\mathbf{y}^{\mathscr{R}} + P_1(1-\mathbf{y}^{\mathscr{R}})\right) \\ & + (1-p)\,\lambda\, u\left(\tfrac{\mathbf{y}^{\mathscr{R}} + P_1(1-\mathbf{y}^{\mathscr{R}})}{\lambda}\right) + (1-p)\,(1-\lambda)\, u\left(\tfrac{R}{\mathbf{P}_2}\tfrac{\mathbf{P}_2 - P_1}{1-\lambda}(1-\mathbf{y}^{\mathscr{R}})\right). \end{aligned} \tag{23}$$

We therefore conclude[9]

Corollary 3 *Suppose relative risk aversion is non-increasing. Comparing any two equilibria in which run-prone banks exist, expected utility is higher in the equilibrium in which the asset price P_1 is higher.*

In equilibria with non-trivial sunspots, no run-prone banks exist and there is real indeterminacy of equilibria if the sunspot probability is above some threshold $\check{p}$. Comparing any two such equilibria, it suffices to consider the expected indirect utility for price combinations for which liquidity supply is zero. For $P_2 = h(P_1)$, this indirect utility is

$$V^{\mathscr{S}}(\mathbf{P}) = \lambda u\left(\frac{P_1}{\lambda P_1 + 1 - \lambda}\right) + (1-\lambda)\, u\left(\frac{R}{\lambda P_1 + 1 - \lambda}\right). \tag{24}$$

Theorem 4 thus leads to the following conclusion.

Corollary 4 *Comparing any two equilibria with non-trivial sunspots in which no run-prone bank exists, expected utility is higher in the equilibrium in which the asset price P_1 is higher.*

[9] See Appendix D.

Given that equilibria differ in terms of expected utilities for consumers and with respect to banks' portfolio choices, a well chosen policy might be able to help consumers to select the most desirable of them. Imposing simple liquidity ratios is a frequently suggested instrument to regulate banks potentially suffering from liquidity problems. However, our analysis suggests to exercise caution.

To back this claim, we consider three liquidity ratios. The first takes aggregate reserves relative to the total amount banks have promised to pay depositors, $\bar{y}/\bar{d}$ with $\bar{y} = \rho y^{\mathscr{R}} + (1-\rho)y^{\mathscr{S}}$ and $\bar{d} = \rho d^{\mathscr{R}} + (1-\rho)d^{\mathscr{S}}$. This measure has the same value $\bar{y}/\bar{d} = \lambda$ in all equilibria. This is because run-proof banks do not speculate on fire sale prices. Therefore, in every equilibrium the banking sector as a whole has just enough reserves to satisfy all impatient consumers provided there is no bank run. This holds regardless which asset prices prevail and how many banks are run-prone.

A simple aggregate reserve ratio, which measures total reserves relative to what banks raise from depositors, $\bar{y} = \rho y^{\mathscr{R}} + (1-\rho)y^{\mathscr{S}}$ is also of only limited usefulness for regulators. This time it is because the relationship between this measure and welfare is non-monotonic. Consider an economy for which a risky banking sector as well as safe banking sectors constitute equilibria, and where the safe banking sector provides higher expected utility (as in Example 2). Then, an aggregate reserve ratio $\bar{y} = \lambda$ (which holds with a safe banking sector and stable asset prices) is associated with a higher expected utility for consumers than a ratio $\bar{y} > y^* > \lambda$ (which holds with a risky banking sector). The former is also associated with a higher expected utility than a ratio $\bar{y} < \lambda$ (which holds with a safe banking sector and volatile prices when sunspots are non-trivial). The problem of non-monotonicity is aggravated by the fact that equilibria with trivial sunspots often also exist where liquidity insurance is efficient and the aggregate reserve ratio is between the one associated with a risky banking sector and the one with a safe banking sector.

The third liquidity ratio is taken from the new Basel Framework which stipulates that the amount of available stable funding has to cover at least 100% of the required stable funding. In the context of our model the required stable funding is given by the share of productive investments, $1-\bar{y}$. The amount of available stable funding are the funds expected to be normally kept in the bank. It is given by what depositors are entitled to withdraw at the interim date $t = 1$ but, provided there is no crisis, do not withdraw from the banking sector. Expressing both in present value terms as of date $t = 0$, this liquidity ratio is given by $\bar{d}(1-\lambda)/(1-\bar{y})$. Since $\bar{y}/\bar{d} = \lambda$ for all equilibria, this ratio is equivalent to $(\bar{y}/(1-\bar{y}))((1-\lambda)/\lambda)$, which is strictly increasing in the amount of aggregate reserves held in the banking sector. It therefore contains the same information as the simple aggregate reserve ratio. Moreover, this ratio is larger than one if and only if aggregate reserves are larger than λ. Therefore, this ratio is strictly larger than one with a risky banking sector and in equilibria with trivial sunspots, but at most one with a safe banking sector and non-trivial sunspots. Ratios larger than one are thus not necessarily an indicator for a safe banking sector but can also indicate that an economy is headed towards a rather wide-spread banking crisis.

5 Concluding remarks

Simultaneous asset market crashes and bank failures can be the result of coordination failures among bank depositors triggered by sunspots. In equilibrium, run-prone banks which expose themselves to such bank runs may exist. There are other types of equilibria in which at least some run-proof banks exist. These banks hold portfolios that take away the incentives for consumers to coordinate on bank runs. Consumption by at least some patient and impatient consumers is stochastic if run-prone banks exist and the financial sector may provide too little liquidity insurance when run-proof banks exist.

The possibility of multiple equilibria, which differ in terms of both, expected utilities and banks' reserve holdings, together with the finding that market liquidity and banks' reserve holding are substitutes, lends itself to the issue of optimal liquidity regulation. However, we leave it for further research to analyze how to design optimum liquidity requirements for economies like those in this paper.

We have considered a rather limited set of options for consumers to interact with banks. A key feature in the world financial crisis has been that funds withdrawn from one bank were re-deposited in another bank. This migration of deposits when banks get into distress is a channel through which the available aggregate liquidity is distributed in times of systemic crises. As this channel would work parallel to, and possibly interacts with, asset markets, the implications of deposit migration on asset prices and the risk-taking behavior of banks in equilibrium remains to be explored.

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A Speculation on fire-sales

This appendix shows that for relative risk aversion $k(x) = -xu''(x)/u'(x) > 1$, run-proof banks do not speculate on buying assets from run-prone banks by holding more reserves than necessary to deter consumers from running. In section 3.5 it has been shown that a necessary condition for run-prone banks and thus a positive supply of productive assets to exist is that prices satisfy $P_1 \leq P_2$. For $P_1 \leq P_2$, suppose the constraint (3) would never be binding. The associated FOC are

$$\begin{aligned} u'(d) &= R\left(u'(x_{2,1})\frac{p}{P_1} + u'(x_{2,2})\frac{1-p}{P_2}\right), \\ u'(x_{2,1}) &= -\frac{1-p}{p}\frac{P_1}{1-P_1}\frac{1-P_2}{P_2}u'(x_{2,2}). \end{aligned}$$

with $x_{2,1} = \frac{R}{P_1}\frac{y+P_1(1-y)-\lambda d}{(1-\lambda)}$ and $x_{2,2} = \frac{R}{P_2}\frac{y+P_2(1-y)-\lambda d}{(1-\lambda)}$. There is a d which maximizes expected utility and satisfies $d < y + P_1(1-y)$ if

$$u'(y+P_1(1-y)) < p\frac{R}{P_1}u'\left(\frac{R}{P_1}(y+P_1(1-y))\right) + (1-p)\frac{R}{P_2}u'\left(\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right).$$

To show that this cannot be, we argue that

$$\frac{R}{P_1}u'\left(\frac{R}{P_1}(y+P_1(1-y))\right) > u'(y+P_1(1-y)), \tag{A1}$$

and

$$\frac{R}{P_2}u'\left(\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right) > u'(y+P_1(1-y)), \tag{A2}$$

cannot be true. Condition (A1) cannot hold for $-\frac{u''(x)}{u'(x)}x > 1$ since

$$\frac{R}{P_1}u'\left(\frac{R}{P_1}(y+P_1(1-y))\right) = u'(y+P_1(1-y)) + \frac{1}{y+P_1(1-y)}\int_{y+P_1(1-y)}^{\frac{R}{P_1}(y+P_1(1-y))}\left[u'(x)+xu''(x)\right]dx.$$

As regards condition (A2), consider first the differential equation

$$\begin{aligned} u'(y+P_1(1-y)) &= \frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)(y+P_1(1-y))}u'\left(\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right) \\ &\quad -\frac{1}{y+P_1(1-y)}\int_{y+P_1(1-y)}^{\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}}\left[u'(x)+xu''(x)\right]dx. \end{aligned}$$

Condition (A2) would hold if

$$u'\left(\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right)\frac{R}{P_2} \quad > \quad \frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)(y+P_1(1-y))}u'\left(\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right)$$
$$-\frac{1}{y+P_1(1-y)}\int_{y+P_1(1-y)}^{\frac{R}{P_2}\frac{(1-\lambda)y+\left(P_2-\lambda P_1\right)(1-y)}{(1-\lambda)}}\left[u'(x)+xu''(x)\right]dx.$$

Rearranging terms gives

$$\frac{R}{P_2}u'\left(\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right)\left(\frac{(P_2-P_1)(1-y)}{(1-\lambda)}\right) < \int_{y+P_1(1-y)}^{\frac{R}{P_2}\frac{(1-\lambda)y+\left(P_2-\lambda P_1\right)(1-y)}{(1-\lambda)}}\left[u'(x)+xu''(x)\right]dx.$$

However, this cannot be for $-\frac{u''(x)}{u'(x)}x > 1$ if $P_1 \leq P_2$.

B Proofs

This appendix contains the formal proofs of our main results.

B.1 Proof of Theorem 1

In order for a bank to be run-proof it needs to be able to pay the relevant depositors at date $t = 1$, i.e.

$$d \leq y + N(P_1)(1-y) \text{ for } s = 1,$$
$$\lambda d \leq y + N(P_2)(1-y) \text{ for } s = 2,$$

and patient depositors are better off withdrawing their funds at date $t = 2$ than at date $t = 1$, i.e.

$$d \leq \frac{M(P_1)}{1-\lambda}(y + N(P_1)(1-y) - \lambda d) \text{ for } s = 1,$$

$$d \leq \frac{M(P_2)}{1-\lambda}(y + N(P_2)(1-y) - \lambda d) \text{ for } s = 2,$$

or equivalently

$$\begin{aligned} d &\leq \frac{M(P_1)}{1-\lambda+\lambda M(P_1)}(y+N(P_1)(1-y)) \text{ for } s=1, \\ d &\leq \frac{M(P_2)}{1-\lambda+\lambda M(P_2)}(y+N(P_2)(1-y)) \text{ for } s=2. \end{aligned}$$

It is easily seen that

$$\begin{aligned} y+N(P_1)(1-y) &\leq \frac{M(P_1)}{1-\lambda+\lambda M(P_1)}(y+N(P_1)(1-y)), \\ y+N(P_2)(1-y) &\geq \frac{\lambda M(P_2)}{1-\lambda+\lambda M(P_2)}(y+N(P_2)(1-y)). \end{aligned}$$

Let the correspondences $B_1, B_2 : \mathbb{R}_{++} \to [0,1] \times \mathbb{R}_+$ be defined by

$$\begin{aligned} B_1(P_1) &= \{ (y,d) \mid d \leq y+N(P_1)(1-y) \}, \\ B_2(P_2) &= \left\{ (y,d) \mid d \leq \frac{M(P_2)}{1-\lambda+\lambda M(P_2)}(y+N(P_2)(1-y)) \right\}. \end{aligned}$$

For the function $b : \mathbb{R}^2_{++} \to \mathbb{R}_+$ defined by

$$b(P_1,P_2) = \max_{y\in[0,1]} \left\{ y+N(P_1)(1-y), \frac{M(P_2)}{1-\lambda+\lambda M(P_2)}(y+N(P_2)(1-y)) \right\},$$

consider the consumer problem

$$\max_{(y,d,x)} \quad \lambda(pu(x_{1,1})+(1-p)u(x_{1,2}))+(1-\lambda)(pu(x_{2,1})+(1-p)u(x_{2,2}))$$

$$\text{s.t.} \quad \begin{cases} \left.\begin{array}{rcl} x_{1,1} & \leq & d \\ x_{2,1} & \leq & \dfrac{M(P_1)}{1-\lambda}(y+N(P_1)(1-y)-\lambda d) \end{array}\right\} \text{for } (y,d) \in B_1(P_1), \\ \left.\begin{array}{rcl} x_{1,1} & \leq & y+N(P_1)(1-y) \\ x_{2,1} & \leq & y+N(P_1)(1-y) \end{array}\right\} \text{for } (y,d) \notin B_1(P_1), \\ \left.\begin{array}{rcl} x_{1,2} & \leq & d \\ x_{2,2} & \leq & \dfrac{M(P_2)}{1-\lambda}(y+N(P_2)(1-y)-\lambda d) \end{array}\right\} \text{for } (y,d) \in B_2(P_2), \\ \left.\begin{array}{rcl} x_{1,2} & \leq & y+N(P_2)(1-y) \\ x_{2,2} & \leq & y+N(P_2)(1-y) \end{array}\right\} \text{for } (y,d) \notin B_2(P_2), \\ \quad y \in [0,1], \\ \quad d \in [0,b(P_1,P_2)]. \end{cases}$$

For all $(P_1,P_2) \in \mathbb{R}^2_{++}$ there is a solution because the set of alternatives is compact. According to Berge's maximum theorem the solution correspondence $F : \mathbb{R}^2_{++} \to [0,1] \times \mathbb{R}_+ \times \mathbb{R}^4_+$ is upper hemi-continuous with non-empty values because expected utility is a continuous function and the set of alternatives is a continuous correspondence.

Let the correspondence $G : \mathbb{R}^2_{++} \to \mathbb{R}^2$ be defined as follows: for $(y,d,x) \in F(P_1,P_2)$ with $(y,d) \in B_s(P_s)$,

$$G_s(P_1,P_2) \;=\; \begin{cases} \left\{ \dfrac{y+\varepsilon(1-y)-\lambda d}{P_s} \right\} & \text{for } P_s < \varepsilon \\ \left[\dfrac{y-\lambda d}{P_s}, \dfrac{y+\varepsilon(1-y)-\lambda d}{P_s} \right] & \text{for } P_s = \varepsilon \\ \left\{ \dfrac{y-\lambda d}{P_s} \right\} & \text{for } \varepsilon < P_s < R \\ \left[-(1-y), \dfrac{y-\lambda d}{P_s} \right] & \text{for } P_s = R \\ \{-(1-y)\} & \text{for } P_s > R \end{cases}$$

for both s; and, for $(y,d,x) \in F(P_1,P_2)$ with $(y,d) \notin B_s(P_s)$,

$$G_s(P_1,P_2) \;=\; \begin{cases} \{0\} & \text{for } P_1 < \varepsilon \\ [-(1-y), 1-y] & \text{for } P_1 = \varepsilon \\ \{-(1-y)\} & \text{for } P_1 > \varepsilon. \end{cases}$$

Then G is upper hemi-continuous.

For $(P_1,P_2) \in \mathbb{R}^2_{++}$ and $(y,d,x) \in F(P_1,P_2)$, if $P_s < \varepsilon$ and $(z_1,z_2) \in G(P_1,P_2)$, then $z_s \geq 0$. For $(P_1,P_2) \in \mathbb{R}^2_{++}$ and $(y,d,x) \in F(P_1,P_2)$, if $P_s > R$ and $(z_1,z_2) \in G(P_1,P_2)$, then $z_s \leq 0$. Therefore prices are bounded from below by $\varepsilon - \delta$ and from above by $R+\delta$ for some $\delta \in]0,\varepsilon[$, $(P_1,P_2) \in [\varepsilon-\delta, R+\delta]^2$.

For $A \subset \mathbb{R}^2$ being the convex hull of the range of G with prices restricted to the set $[\varepsilon-\delta, R+\delta]^2$,

$$A \;=\; \mathrm{co}\{\, (z_1,z_2) \in \mathbb{R}^2 \mid \exists (P_1,P_2) \in [\varepsilon-\delta, R+\delta]^2 : (z_1,z_2) \in G(P_1,P_2) \,\}$$

let the correspondence $H : A \to [\varepsilon-\delta, R+\delta]^2$ be defined by

$$H(z_1,z_2) \;=\; \{\, (P_1,P_2) \in [\varepsilon-\delta, R+\delta]^2 \mid \forall (P_1',P_2') \in [\varepsilon-\delta, R+\delta]^2 : P_1 z_1 + P_2 z_2 \geq P_1' z_1 + P_2' z_2 \,\}.$$

Then H is upper hemi-continuous.

The correspondence $(\text{co}\,G, H) : [\varepsilon - \delta, R + \delta]^2 \times A \to [\varepsilon - \delta, R + \delta]^2 \times A$ has a fixed point according to Kakutani's fixed point theorem, because $[\varepsilon - \delta, R + \delta]^2 \times A$ is convex and compact and $(\text{co}\,G, H)$ is convex valued and upper hemi-continuous. Suppose $(P_1, P_2, z_1, z_2) \in [\varepsilon - \delta, R + \delta]^2 \times A$ is a fixed point, so $(z_1, z_2) \in \text{co}\,G(P_1, P_2)$ and $(P_1, P_2) \in H(z_1, z_2)$. Suppose $z_s \neq 0$, then $H_s(z_1, z_2) = \varepsilon - \delta$ in case $z_s < 0$ and $H_s(z_1, z_2) = R + \delta$ in case $z_s > 0$. Suppose $P_s = \varepsilon - \delta$, then either $z_s = 0$ or $z_s > 0$ contradicting $P_s = \varepsilon - \delta$, so $z_s = 0$. If $P_s = R + \delta$, then either $z_s = 0$ or $z_s < 0$ contradicting $P_1 = R + \delta$, so $z_s = 0$. Therefore $z_s = 0$ for both s.

For every $(z_1, z_2) \in \text{co}\,G(P_1, P_2)$ there are at most three points $(z_1^i, z_2^i)_i$ with $(z_1^i, z_2^i) \in G(P_1, P_2)$ for every i and at most three weights $(w^i)_i$ with $w^i > 0$ for every i and $\sum_i w^i = 1$ such that $(z_1, z_2) = \sum_i w^i (z_1^i, z_2^i)$ according to Caratheodory's theorem. Hence (P_1, P_2, z_1, z_2) is an equilibrium.

B.2 Proof of Theorem 2

Because $P_2 \geq \varepsilon$, we have $\varepsilon \leq (1-p)/(1-p/P_1)$. Together with $P_1 \geq (1/\lambda - 1)(1/y^* - 1)^{-1}$, there is thus an upper bound for p given by $p^T := (1-\varepsilon)/(1-\varepsilon \frac{\lambda}{1-\lambda} \frac{1-y^*}{y^*})$. $p^T > 0$ because $\varepsilon < 1$ and $p^T < 1$ because $\lambda < y^*$ for relative risk aversion larger one. Run-prone banks have no incentive to enter the market because the allocation obtained by run-proof banks is the solution to the unconstrained problem. Being subjected to the additional constraints associated with a failure in one of the extrinsic states would imply that run-prone banks offer less than the first-best expected utility.

B.3 Proof of Theorem 3

$\rho = 0$ requires $q^S = 0$. Absence of asset price volatility requires $P_1 = P_2 = 1$. For run-proof banks, the budget constraints (4) then imply $x_{1,1}^{\mathscr{S}} = x_{1,2}^{\mathscr{S}} = d^{\mathscr{S}} = 1$, $x_{2,1}^{\mathscr{S}} = x_{2,2}^{\mathscr{S}} = R$ and $y^{\mathscr{S}} = \lambda$. For run-prone banks, $P_1 = P_2 = 1$ implies $x_{1,1}^{\mathscr{R}} = x_{2,1}^{\mathscr{R}} = 1$ while $d^{\mathscr{R}}$ solves

$$u'(d^{\mathscr{R}}) = Ru'\left(R\frac{1-\lambda d^{\mathscr{R}}}{1-\lambda}\right),$$

implying $x^{\mathscr{R}}_{1,2} = d^{\mathscr{R}} = y^*/\lambda$ and $x^{\mathscr{R}}_{2,2} = R(1-\lambda d^{\mathscr{R}})/(1-\lambda) = R(1-y^*)/(1-\lambda)$. Let

$$X(p) = (1-p)\,\lambda u\left(\tfrac{y^*}{\lambda}\right) + (1-p)\,(1-\lambda)\,u\left(\tfrac{R(1-y^*)}{1-\lambda}\right) + pu(1),$$

and $\check{p}$ be a solution to

$$\lambda u(1) + (1-\lambda)\,u(R) = X(p).$$

Note that y^* maximizes expected utility in absence of sunspots. Therefore, $\lambda u(1) + (1-\lambda)\,u(R) < \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$. $u' > 0$ implies $u(1) < \lambda u(1) + (1-\lambda)\,u(R)$. Since $X' < 0$, there is a unique $\check{p} < 1$ such that $V^{\mathscr{S}}(\mathbf{P}) \geq V^{\mathscr{R}}(\mathbf{P})$ for $\mathbf{P} = (1,1)$ if and only if $p \geq \check{p}$.

B.4 Proof of Theorem 4

$\rho = 0$ requires $P_2 = h(P_1)$. Continuity of h implies there exists a continuum of equilibrium prices which support equilibria with safe banking sectors provided $V^{\mathscr{S}}(1,1) > V^{\mathscr{R}}(1,1)$. In an arbitrage-free equilibrium, $P_2 \leq R$. Hence, P_1 is strictly bounded away from ε since $h^{-1}(R) > 0$.

Indirect utility is given by

$$V^{\mathscr{S}}(\mathbf{P}) = \lambda u\left(\frac{P_1}{\lambda P_1 + 1 - \lambda}\right) + (1-\lambda)\,u\left(\frac{R}{\lambda P_1 + 1 - \lambda}\right),$$

With $P_2 = h(P_1)$, applying the Envelope theorem yields

$$\begin{aligned}\frac{dV^{\mathscr{S}}(\mathbf{P})}{dP_1} &= \lambda u'\left(\frac{P_1}{\lambda P_1 + 1 - \lambda}\right)\frac{1-\lambda}{(\lambda P_1 + 1 - \lambda)^2} \\ &\quad - (1-\lambda)\,u'\left(\frac{R}{\lambda P_1 + 1 - \lambda}\right)\frac{\lambda R}{(\lambda P_1 + 1 - \lambda)^2}.\end{aligned}$$

Since $k(x) > 1$ implies $y^*/\lambda > 1$ and thus $\frac{1-\lambda}{\lambda}\frac{y^*}{1-y^*} > 1$, it follows $dV^{\mathscr{S}}(\mathbf{P})/dP_1 > 0$ for all $P_1 \in \left[h^{-1}(R), \frac{1-\lambda}{\lambda}\frac{y^*}{1-y^*}\right]$ because $u'(x) \geq Ru'\left(\frac{R(1-\lambda x)}{1-\lambda}\right)$ for all $x \leq y^*/\lambda$ (since $k(x) > 1$) and $\frac{d}{dP_1}(u'(\frac{P_1}{\lambda P_1 + 1 - \lambda}) - Ru'(\frac{R}{\lambda P_1 + 1 - \lambda})) < 0$ (since $u'' < 0$) together imply $u'(\frac{P_1}{\lambda P_1 + 1 - \lambda}) \geq Ru'(\frac{R}{\lambda P_1 + 1 - \lambda})$

for all $P_1 \in \left[h^{-1}(R), \frac{1-\lambda}{\lambda}\frac{y^*}{1-y^*}\right]$. Finally, according to Theorem 3, $y^{\mathscr{S}} = \lambda$ for $P_1 = 1$. Since $y^{\mathscr{S}} = \lambda P_1/(\lambda P_1 + 1 - \lambda)$ with $\frac{d}{dP_1}\lambda P_1/(\lambda P_1 + 1 - \lambda) > 0$, we have $y^{\mathscr{S}} < \lambda$ for $P_1 < 1$.

B.5 Proof of Lemma 1

$\rho = 1$ implies $Q^S = 0$. Accordingly, for equilibria with $\rho = 1$ it requires $\lambda d^{\mathscr{R}} - y^{\mathscr{R}} = 0$ and either $1 - y^{\mathscr{R}} = 0$ or $P_1 \leq \varepsilon$. We rule out $1 - y^{\mathscr{R}} = 0$ because state-independence of liquidity demand requires $y^{\mathscr{R}}$ to solve

$$\frac{u'\left(\frac{y^{\mathscr{R}} + P_1(1-y^{\mathscr{R}})}{\lambda}\right)}{u'\left(\frac{R}{P_2}\frac{P_2-P_1}{1-\lambda}(1-y^{\mathscr{R}})\right)} - \frac{R}{P_2} = 0,$$

and concavity of u implies $y^{\mathscr{R}} \leq \lambda R/(\lambda R + 1 - \lambda) < 1$. Hence, an equilibrium exists only if $P_1 \leq \varepsilon$ and $f(\varepsilon) \neq \emptyset$, i.e. there is some $(y^{\mathscr{R}}, P_2) \in [0, \lambda R/(\lambda R + 1 - \lambda)] \times [1, R]$ satisfying

$$\begin{aligned}
\frac{u'\left(y^{\mathscr{R}}/\lambda\right)}{u'\left(\frac{R(1-y^{\mathscr{R}})}{1-\lambda}\right)} &= \frac{R}{P_2}, \\
\frac{u'\left(y^{\mathscr{R}}\right)}{u'\left(\frac{R(1-y^{\mathscr{R}})}{1-\lambda}\right)} &= \frac{R}{P_2}\frac{1-p}{p}(P_2 - 1).
\end{aligned}$$

Let Y_1 be the solution to the first equation for a given P_2. Then, $\lim_{P_2 \to 1} Y_1 = y^*$, $\lim_{P_2 \to R} Y_1 = \lambda R/(\lambda R + (1-\lambda))$ and $dY_1/dP_2 > 0$. Let Y_2 be the solution to the second equation for a given P_2. Then, $\lim_{P_2 \to 1} Y_2 = 1$, $\lim_{P_2 \to R} Y_2 = \tilde{y} \in (0,1)$ and $dY_2/dP_2 < 0$ where $\tilde{y}$ is implicitly defined by

$$\frac{u'(\tilde{y})}{u'\left(\frac{R(1-\tilde{y})}{1-\lambda}\right)} = \frac{1-p}{p}(R-1).$$

Since $y^* < 1$, there is no $f(\varepsilon) \in [0, \lambda R/(\lambda R + 1 - \lambda)] \times [1, R]$ if

$$\frac{u'\left(\frac{\lambda R}{\lambda R + (1-\lambda)}\right)}{u'\left(\frac{R}{\lambda R + (1-\lambda)}\right)} > \frac{1-p}{p}(R-1),$$

or, equivalently, if $p > \hat{p}$.

B.6 Proof of Theorem 5

According to Lemma 1, provided $p \leq \hat{p}$ there is some $(y^{\mathscr{R}}, P_2) \in [0, \lambda R/(\lambda R + 1 - \lambda)] \times [1, R]$ for which liquidity demand in either state is zero. By the implicit function theorem, (12a) and (12b) imply for $P_1 = \varepsilon$ that $\lim_{p \to 0} P_2 = 1$ and $\lim_{p \to 0} y^{\mathscr{R}} = y^*$. Therefore, for $P_1 = \varepsilon$,

$$\lim_{p \to 0} V^{\mathscr{R}}(\mathbf{P}) = \lambda u \left(\frac{y^*}{\lambda} \right) + (1 - \lambda) u \left(R \frac{1 - y^*}{1 - \lambda} \right).$$

For $P_1 = \varepsilon$ and $p \to 0$ the first-order condition for run-proof banks becomes

$$u'(y) \leq R u' \left(R \frac{1 - \lambda y}{1 - \lambda} \right),$$

which would hold with equality only if some $y \in (0,1)$ were a solution. However, since $k(x) > 1$, there is no $y \in (0,1)$ to meet the first-order condition with equality. Hence, $y^{\mathscr{S}} = 1$ which implies

$$\lim_{p \to 0} V^{\mathscr{S}}(\mathbf{P}) = \lambda u(1) + (1 - \lambda) u(R).$$

$k(x) > 1$ further implies $\lim\limits_{p \to 0} V^{\mathscr{R}}(\mathbf{P}) > \lim\limits_{p \to 0} V^{\mathscr{S}}(\mathbf{P})$. Therefore, provided $P_1 = \varepsilon$ and $q_1^D = q_2^D = 0$, either is $V^{\mathscr{R}}(\mathbf{P}) > V^{\mathscr{S}}(\mathbf{P})$ for all $p \leq \hat{p}$, or by the intermediate value theorem there is a $\bar{p} \leq \hat{p}$ such that $V^{\mathscr{R}}(\mathbf{P}) > V^{\mathscr{S}}(\mathbf{P})$ for all $p < \bar{p}$. The equilibrium is locally isolated because for $p < \bar{p}$ the solution to the bank's problem, satisfying (16a) and (16b), is unique. (16a) implies $y^{\mathscr{R}} > y^*$ and thus $V^{\mathscr{R}}(\mathbf{P}) < \lambda u(y^*/\lambda) + (1 - \lambda) u(R(1 - y^*)/(1 - \lambda))$.

B.7 Proof of Theorem 6

$q_1^D = q_2^D \geq 0$ and thus $d = (y + P_1(1 - y))/\lambda$ hold in any equilibrium with $\rho \in]0,1]$. For a given $P_2 \in [1, R]$, a necessary condition is that there is a $(P_1, y) \in [\varepsilon, 1] \times [0, 1]$ such that condition (12a) is met. If there is such a pair, it satisfies $dy/dP_1 < 0$. Note, if $R < \lambda^{-1}$ there is no $P_2 \in [1, R]$ such that liquidity demand is state-independent for $P_1 = 1$. Condition (12b) reads

$$(1-P_1)\frac{u'\left(y+P_1(1-y)\right)}{u'\left(\frac{R}{P_2}\frac{y+P_2(1-y)-\lambda d}{1-\lambda}\right)} \leq (P_2-1)\frac{1-p}{p}\frac{R}{P_2}.$$

The right side converges to 0 if $p \to 1$. The marginal rate of substitution in condition (12b) converges to $u'(1)/u'\left(\frac{R}{P_2}\frac{(P_2-1)(1-y)}{1-\lambda}\right) > 0$ if $P_1 \to 1$, where y is either zero or satisfies

$$\frac{u'(1/\lambda)}{u'\left(\frac{R}{P_2}\frac{(P_2-P_1)(1-y)}{1-\lambda}\right)} = \frac{R}{P_2}.$$

Therefore, if $p \to 1$ then either P_1 converges to 1 for a given $P_2 \in [1,R]$, or liquidity demand cannot be state-independent.

As for liquidity supply, note that $\lim_{p\to 1} h^{-1}(P_2) = 1$ for all $P_2 \in [1,R]$. Therefore, if $p \to 1$ and $P_1 \to 1$, $q^S \geq 0$ for all $P_2 \in [1,R]$. Provided $q_1^D = q_2^D \geq 0$ for $p \to 1$ and $P_1 \to 1$, $V^{\mathscr{R}}(\mathbf{P})$ converges to $u(1)$ while $V^{\mathscr{S}}(\mathbf{P})$ converges to $\lambda u(1) + (1-\lambda)u(R) > u(1)$. However, if liquidity demand cannot be state-independent, run-prone banks cannot exist anyway whilst $q^S = 0$.

Therefore, either there is no $\mathbf{P} \in [\varepsilon,1] \times [1,R]$ for which $q^S \geq 0$, $q_1^D = q_2^D \geq 0$ and $V^{\mathscr{S}}(\mathbf{P}) \leq V^{\mathscr{R}}(\mathbf{P})$ for all $p \geq \check{p}$. Or, if there is some $p > \check{p}$ for which some $\mathbf{P} \in [\varepsilon,1] \times [1,R]$ exists such that $q^S \geq 0$, $q_1^D = q_2^D \geq 0$ and $V^{\mathscr{S}}(\mathbf{P}) \leq V^{\mathscr{R}}(\mathbf{P})$, then there is some $\tilde{p} \in]\check{p}, 1[$ such that for all $p > \tilde{p}$ there is no $\mathbf{P}$ for which $q_1^D = q_2^D \geq 0$ and $V^{\mathscr{S}}(\mathbf{P}) \leq V^{\mathscr{R}}(\mathbf{P})$ according to the intermediate value theorem.

B.8 Proof of Theorem 7

Again, $q_1^D = q_2^D \geq 0$ and thus $d = (y + P_1(1-y))/\lambda$ hold in any equilibrium with $\rho \in]0,1]$. Condition (12b) reads

$$p\frac{u'\left(y+P_1(1-y)\right)}{u'\left(\frac{R}{P_2}\frac{y+P_2(1-y)-\lambda d}{1-\lambda}\right)} \leq (1-p)\frac{P_2-1}{1-P_1}\frac{R}{P_2},$$

with strict inequality only if $y = 0$. The left hand side converges to zero for $p \to 0$, whereas the right hand side converges to $\frac{P_2-1}{1-P_1}\frac{R}{P_2} > 0$. Hence, as long as $y^{\mathscr{R}} > 0$ such that above condition holds with equality, it follows for a given P_1 that $P_2 \to 1$.

Provided $P_2 \to 1$ and $P_1 \in [\varepsilon, y^*[$, condition (12a) implies $x^{\mathscr{R}}_{1,2} = y^*/\lambda$, $x^{\mathscr{R}}_{2,2} = R(1-y^*)/(1-\lambda)$, $y^{\mathscr{R}} = (y^* - P_1)/(1-P_1) > 0$, and $V^{\mathscr{R}}(\mathbf{P}) = \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$. For $P_2 \to 1$ and $P_1 \in [\varepsilon, y^*[$, run-proof banks optimally store $y^{\mathscr{S}} = \max\{1, (y^*/\lambda - P_1)/(1-P_1)\} = 1$ such that $V^{\mathscr{S}}(\mathbf{P}) = \lambda u(1) + (1-\lambda)u(R) < \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$.

Concavity of u together with the budget constraints (7) imply that the left side in (12a) is a continuous, monotone and decreasing function of $y^{\mathscr{R}}$ and continuous, monotone and increasing in P_2. Hence, for $y^{\mathscr{R}} = 0$, there is at most one P_2 satisfying (17). The projection ϕ_1 of f on the P_2-coordinate provided $y^{\mathscr{R}} = 0$ is a bijective function $\phi_1 : [\phi_1^{-1}(1), \min\{1, \lambda R\}] \times [1, \min\{R, \phi_1(1)\}]$ with

$$\frac{dP_2}{dP_1} = \frac{k_{2,2} + \left(\frac{P_2}{P_1} - 1\right) k_{1,1}}{k_{2,2} + \left(\frac{P_2}{P_1} - 1\right)} \frac{P_2}{P_1} > 0,$$

where $k_{t,s} = k(x^{\mathscr{R}}_{t,s})$ is relative risk aversion at $x^{\mathscr{R}}_{t,s}$. For $\rho \in]0,1[$ it must be that $V^{\mathscr{R}}(\mathbf{P}) = V^{\mathscr{S}}(\mathbf{P})$. However, according to (18a) and (18c), $V^{\mathscr{R}}(\mathbf{P}) > \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$. Hence, $V^{\mathscr{R}}(\mathbf{P}) > V^{\mathscr{S}}(\mathbf{P})$. Therefore, $\rho \in]0,1[$ cannot be an equilibrium.

Finally, according to Theorem 4, $V^{\mathscr{S}}(\mathbf{P}) \leq \lambda u(1) + (1-\lambda)u(R) < \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$ for all $P_2 = h(P_1)$. Since (i) $\phi^{-1}(P_2) \leq h^{-1}(P_2)$ for $\phi^{-1}(P_2) \neq \emptyset$, (ii) $V^{\mathscr{R}}(\mathbf{P}) \geq \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$ for $P_1 = \phi^{-1}(P_2)$, and (iii) $dV^{\mathscr{R}}(\mathbf{P})/dP_1 > 0$ we have $V^{\mathscr{R}}(\mathbf{P}) > \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$. Hence, $\rho = 0$ cannot be an equilibrium.

C State-independent liquidity demand

This appendix shows that non-increasing relative risk aversion is a sufficient condition that all combinations of asset prices for which liquidity demand is state-independent is described by a continuous function that maps P_1 onto P_2. For any $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2)$, equation (17) defines P_2 and $y^{\mathscr{R}}$

as implicit functions of P_1 in some neighborhood of $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2)$ according to the general implicit function theorem. Provided $y^{\mathscr{R}} \in]0,1[$, each of these solutions satisfy

$$\frac{dP_2}{dP_1} = -\frac{(k_{1,1}-k_{1,2})\,k_{2,2}\frac{P_2-1}{P_2-P_1} + k_{1,2} + k_{2,2}\frac{y^{\mathscr{R}}+P_1(1-y^{\mathscr{R}})}{(1-P_1)(1-y^{\mathscr{R}})}}{(k_{1,1}-k_{1,2})\,k_{2,2}\frac{P_1}{P_2-P_1} + k_{1,2}\frac{1}{P_2-1} + k_{2,2}\frac{y^{\mathscr{R}}+P_1(1-y^{\mathscr{R}})}{(1-P_1)(1-y^{\mathscr{R}})}\frac{P_2}{P_2-1} + k_{1,1}}\frac{P_2}{1-P_1}$$

and

$$\frac{dy^{\mathscr{R}}}{dP_1} = -\frac{(k_{1,1}-k_{1,2})\,k_{2,2}\frac{P_1}{P_2-P_1} + k_{1,2}\frac{1}{P_2-1} + k_{2,2}\frac{y^{\mathscr{R}}+P_1(1-y^{\mathscr{R}})}{(1-P_1)(1-y^{\mathscr{R}})}\frac{1}{P_2-1} + k_{1,1} + \frac{1}{1-P_1}}{(k_{1,1}-k_{1,2})\,k_{2,2}\frac{P_1}{P_2-P_1} + k_{1,2}\frac{1}{P_2-1} + k_{2,2}\frac{y^{\mathscr{R}}+P_1(1-y^{\mathscr{R}})}{(1-P_1)(1-y^{\mathscr{R}})}\frac{P_2}{P_2-1} + k_{1,1}}\frac{1-y^{\mathscr{R}}}{1-P_1}.$$

For any P_1, equation (12a) defines P_2 as a monotone and increasing function of $y^{\mathscr{R}}$. Then, a sufficient condition that there is at most one $(y^{\mathscr{R}}, P_2)$ satisfying (17) and $y^{\mathscr{R}} > 0$ is that the left side in (12b) is strictly monotone in $y^{\mathscr{R}}$ while taking into account the relation between $y^{\mathscr{R}}$ and P_2 according to (12a). Let

$$\Phi_1 := \left(\frac{k_{1,2}}{k_{2,2}}\frac{1}{P_1} + \left(\frac{y^{\mathscr{R}}}{1-y^{\mathscr{R}}} + P_1\right)\frac{1}{1-P_1}\frac{P_2}{P_1} + \frac{k_{1,1}}{k_{2,2}}\frac{P_2-1}{P_1}\right)\frac{P_2-P_1}{P_2-1},$$

$$\Phi_2 := \left(\frac{k_{1,2}}{k_{2,2}} + \left(\frac{y^{\mathscr{R}}}{1-y^{\mathscr{R}}} + P_1\right)\frac{1}{1-P_1}\right)\frac{P_2-P_1}{P_2-1}.$$

This monotonicity holds if for all P_1 either $\Phi_1 > k_{1,2} - k_{1,1}$ or $\Phi_1 < k_{1,2} - k_{1,1}$. The sign of dP_2/dP_1 is positive if and only if $\Phi_1 > k_{1,2} - k_{1,1} > \Phi_2$. Hence, with non-increasing risk aversion, i.e. $k_{1,1} \geq k_{1,2}$, the projection ϕ_2 of f on the P_2-coordinate provided $y^{\mathscr{R}} \in]0,1[$ is a bijective function $\phi_2 : [\max\{\varepsilon, \phi_2^{-1}(R)\}, \min\{\phi_1^{-1}(1), \phi_2^{-1}(1)\}] \times [1,R]$ satisfying $d\phi_2(P_1)/dP_1 < 0$. Hence, for $P_2 = \phi_2(P_1)$ we have $q_1^D = q_2^D$ and $y^{\mathscr{R}} > 0$. Similarly, the projection of f on $y^{\mathscr{R}}$ satisfies $dy^{\mathscr{R}}/dP_1 < 0$ for $k_{1,1} \geq k_{1,2}$.

Continuity of the projection of f on P_2 holds because (12a) implies that $\phi_1(P_1) = 1$ for some $P_1 \in]0,1[$, where ϕ_1 is the projection of f on the P_2-coordinate provided $y^{\mathscr{R}} = 0$ as defined in the proof of Theorem 7. Moreover, (12a) and (12b) imply that $\phi_2(P_1) > 1$ for all $P_1 \in]0,1[$. Hence, there is a unique $P_1 \in]0,1[$ such that $\phi_1(P_1) = \phi_2(P_1)$ and $\phi_1(P_1) \in]1,R]$.

D Indirect utility and asset prices

This appendix derives the condition under which the indirect utility consumers get in equilibria in which run-prone banks exist is strictly increasing in P_1. Consider indirect utility as given in equation (23). With $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2) = f(P_1)$, applying the Envelope theorem yields

$$\frac{dV^{\mathscr{R}}(\mathbf{P})}{dP_1} = \begin{cases} \dfrac{\left(\frac{k_{2,2}}{1-P_1}+\frac{k_{1,1}}{P_1}+\frac{\mathbf{P}_2-1}{1-P_1}\frac{1}{P_1}\right)(1-p)\left(1-\mathbf{y}^{\mathscr{R}}\right)(\mathbf{P}_2-P_1)u'\left(x_{2,2}^{\mathscr{R}}\right)}{k_{2,2}+\frac{\mathbf{P}_2-P_1}{P_1}} & \text{for} \quad \mathbf{y}^{\mathscr{R}} = 0, \\[2em] \dfrac{(1-p)\left(1-\mathbf{y}^{\mathscr{R}}\right)\left(k_{1,2}+\left(k_{2,2}\left(\frac{\mathbf{y}^{\mathscr{R}}}{1-\mathbf{y}^{\mathscr{R}}}+P_1\right)\frac{1}{1-P_1}+k_{1,1}\right)\frac{\mathbf{P}_2-1}{1-P_1}\right)u'\left(x_{2,2}^{\mathscr{R}}\right)}{(k_{1,1}-k_{1,2})k_{2,2}\frac{P_1}{\mathbf{P}_2-P_1}+k_{1,2}\frac{1}{\mathbf{P}_2-1}+k_{2,2}\left(\frac{\mathbf{y}^{\mathscr{R}}}{1-\mathbf{y}^{\mathscr{R}}}+P_1\right)\frac{1}{1-P_1}\frac{\mathbf{P}_2}{\mathbf{P}_2-1}+k_{1,1}} & \text{for} \quad \mathbf{y}^{\mathscr{R}} > 0. \end{cases}$$

For $\mathbf{y}^{\mathscr{R}} = 0$ we have $dV^{\mathscr{R}}(\mathbf{P})/dP_1 > 0$. For $\mathbf{y}^{\mathscr{R}} > 0$ it is positive if and only if

$$\left(\frac{k_{1,2}}{k_{2,2}}\frac{1}{P_1} + \left(\frac{\mathbf{y}^{\mathscr{R}}}{1-y^{\mathscr{R}}} + P_1\right)\frac{1}{1-P_1}\frac{\mathbf{P}_2}{P_1} + \frac{k_{1,1}}{k_{2,2}}\frac{\mathbf{P}_2-1}{P_1}\right)\frac{\mathbf{P}_2-P_1}{\mathbf{P}_2-1} > k_{1,2} - k_{1,1},$$

for which a sufficient condition is non-increasing relative risk aversion.