Capital Ideas: Optimal Capital Reserve Strategies for a Bank and its Regulator

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Abstract

We formulate a continuous-time model of a deposit taking bank, operating subject to capital adequacy regulation, and where the bank's loans are exposed to default risk. The bank maximises their market value of equity by appropriately controlling loan and equity issuance, dividend payments, and endogenous closure. We show how the bank responds to the span of capital adequacy requirements and to changes in default variance. Of interest to regulators, we also find the capital requirement for which the probability of early closure (through either insolvency or endogenous closure) is minimised, and a level at which lending is maximised. While the bank's optimal capital structure varies nonmonotonically with default variance, the level of capital requirement that minimises the probability of early closure and maximises lending is fairly robust to changes in the bank's default variance/risk; contributing to the ongoing discussion on optimal bank capital.

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1. Introduction

A capital adequacy ratio is a regulatory control which specifies the minimum amount of shareholders' equity a bank must hold in relation to the size of their risky assets. Setting the capital adequacy ratio is a pre-emptive measure designed to ensure a bank can better absorb potential future losses and hence to reduce the likelihood of a bank becoming insolvent; at which point the bank would be placed in resolution so as to help protect depositors, taxpayers, and the stability of the broader financial system (European Commission, 2014c; U.S. Senate, 2010). Yet, despite

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this regulatory control, many banks found themselves significantly under-capitalised during the 2007/08 financial crisis. Such under-capitalisation has led regulators to significantly increase their capital requirements (European Commission, 2014a) and to put in place swift resolution arrangements (e.g., Bank of England, 2017). However, the new capital adequacy levels are much higher than any that have been implemented in recent history, and as such, data on the response of bank behaviour to these higher capital requirements are scarce. Naturally, this has led to much debate on the likely consequences of tougher regulation. Industry groups have warned that the new capital regime could have a severe impact on the economy, with banks potentially reducing lending, reducing dividends, and/or exiting certain lines of business (e.g., Financial Times, 2016, 2013). Others have suggested that high capital requirements will have little negative effect on the economy, thus leaving clear the path to extremely high capital requirements (see Admati, 2014, and references therein).[1]

Motivated by the above points, this paper aims to provide some theoretical insight into the likely response of banks to increased capital requirements, thereby helping regulators to assess the expected costs and benefits of implementing tougher capital requirements.[2] Specifically, we develop and solve a general model of a deposit-taking bank constrained by a capital adequacy ratio. In considering a broad range of capital adequacy ratios we are able to examine their effect on bank insolvency (associated with lower capital requirements and excessive bank leverage) and to show how such capital requirements can lead to *endogenous* bank closure (associated with higher capital requirements and lower bank profitability). Within our results we give particular attention to the response of bank lending levels and to the total probability of bank closure. In considering this latter metric, we show that a finite capital adequacy ratio exists for which the probability of bank closure is minimised; emphasising the need for regulators to strike an informed balance between the costs and benefits of increased capital requirements in order to protect the stability of the financial system.

Acknowledging that any insight gained is only as good as the model one builds, we attempt to accurately model the behaviour of a bank by endogenizing the bank's primary operating and financing decisions/controls. Furthermore, since such decisions are often taken dynamically and are interrelated, we set our model in continuous time. This setting also allows us to investigate the interaction between a bank's behaviour and their loan-loss dynamics. To expand: Whilst operational, we assume the bank seeks to maximise their market value of equity (e.g., Glasserman and Nouri, 2012; Grossman and Stiglitz, 1977), which it achieves by controlling four fundamental business decisions. The first decision is when to dynamically re-invest profits into new loan issuances or safe (cash) investments, thus determining the portfolio partition between the two asset classes (e.g., Estrella, 2004; Tobin, 1982). The second, related, decision we consider is when to pay dividends and how much they should be (Avanzi, 2009; van den Heuvel, 2007; Jeanblanc-

[1] Admati and co-authors argue that bank equity is not expensive. This conclusion is based, in part, on the classical argument that, whilst leverage decisions affect how risk and reward are divided between equity and debt, such decisions do not affect the bank's total funding cost and hence should not impact bank profitability or lending (Modigliani and Miller, 1958). However, market frictions and asymmetric information about a bank's net worth suggest that, in practice, new bank equity is expensive, particularly when it is needed the most (Bolton and Freixas, 2006).

[2] It is hoped that this will be complimentary to existing practices, as regulators have historically used a trial and error approach to the setting of such capital requirements (Allen and Gale, 2003).

Picqué and Shiryaev, 1995). Here the bank must balance paying out dividends, which can then be considered safe assets, against the possibility of growing their equity base, which would be continually exposed to the bank's insolvency risk (Titman and Tsyplakov, 2007).[3] At the other end of the profitability spectrum lies two more bank controls: the option to raise new equity should the bank's equity level become too low (Bolton, Chen, and Wang, 2011), and the option to close down the bank (or a division thereof) should expected profits decline too far (Calveras, 2003). The ability of a bank to voluntarily raise new equity is an important feature in our model, since this allows the bank to self-insure against future loan-loss uncertainty; without its inclusion one would be unfairly ignoring a bank's key line of defence against insolvency. The endogenous closure decision is taken when a firm's valuation of future operations no longer exceeds its current liquidation value, and it is well studied within the real options literature, e.g., Dixit and Pindyck (1994); Brennan and Schwartz (1985). However, its inclusion within dynamic banking models appears much rarer. Indeed, the combining and optimising of these four key business decisions (loan and equity issuance, dividend payment and endogenous closure) in continuous time is where a significant portion of this paper's novelty lies.

To incorporate regulatory control into our modelling, we assume a solvent bank will be forced to issue new equity should they become under-capitalised; and if the bank were to ever become insolvent, we assume the regulator winds the bank up (or a division thereof) in an orderly resolution (for a full explanation see U.S. Senate, 2010; European Commission, 2014c; Bank of England, 2017). We also assume the bank operates optimally in the presence of this given and fixed capital regulation. In doing so, we are able to draw out various economic insights and metrics that span the spectrum of capital adequacy implications, which, we repeat, is particularly pertinent in this high capital ratio, low data, environment.

Unsurprisingly, there is a large literature addressing the effects of bank capital regulations (see, for example, Bank for International Settlements, 2019; Firestone, Lorenc, and Ranish, 2019; Dagher et al., 2016; Posner and Weyl, 2014; Thakor, 2014; Calomiris, 2013; Miles, Yang, and Marcheggiano, 2013, and the references therein). However, there appears to be only limited work in the continuous-time setting. In this regard, our model is perhaps most closely related to the paper of Hugonnier and Morellec (2017), who develop an alternative dynamic banking model to assess the effects of various liquidity and capital requirements on a bank's insolvency risk. However, while providing many valuable insights with their model, we note that since the amount of risky assets held by the bank is fixed, the impact of capital regulation on bank lending (a key consideration of this paper) cannot be adequately assessed in such a model.

The insights that we gain from our model and subsequent analysis are as follows. For low capital adequacy requirements we find that the bank's behaviour (i.e., bank value, loan provisioning, and the probability of bank closure) is largely insensitive to such requirements. This is because it is already optimal for banks to self-insure at those levels. However, as the capital requirements increase beyond those of optimal self-insurance, a larger equity buffer has the clear effect of re-

[3]The importance of this control is illustrated by the findings of Cohen and Scatigna (2016), who report that the retention of earnings (via a reduction of dividend payouts) accounted for almost two-thirds of the increase in bank capital from the end of 2009 to the end of 2012 (based on a sample of 101 large global banks). De Jonghe and Öztekin (2015) also report similar findings.

ducing the likelihood of insolvency: for our base-case parameter values, we find the probability of insolvency can be reduced by an order of magnitude when the capital ratio is increased towards the 15% level. However, this risk reduction only works up to a point, since increasing capital requirements beyond this point significantly increases the likelihood of endogenous closure—the bank's optimal response to a reduced return on equity at these higher capital adequacy requirements.[4] The convex nature of the probability of bank closure (through either insolvency or endogenous closure) against capital requirements demonstrates a minimum probability exists, and we find its existence robust to parameter variations. One may therefore consider the capital requirement that minimises the closure risk as optimal (subject to other regulatory objectives). Indeed, we believe this notion of risk-minimisation may appeal to public-facing policy makers who must weigh up arguments from interested parties in a defensible and transparent fashion.

A large focus of the debate on increased capital requirements centres upon bank lending responses (Bridges et al., 2014). In this regard, we use our model to show that the bank's optimal lending levels have a nonmonotonic response to capital requirements. Up to a point it will be optimal for some banks to increase lending as capital requirements increase, and beyond that they will decrease lending back down as they start to retain earnings as cash rather than lend it out. Economically, this point corresponds to the level of regulation beyond which the bank will optimally start to hold cash (rather than issue new loans) to offset the cost of increased equity requirements (i.e., the marginal value of loans decreases below the marginal value of cash). Furthermore, we observe that the nonmonotonic lending behaviour is evident even when variance is removed from our model, indicating that this response is not driven by the stochastic nature of our model, but rather our assumptions regarding the bank's cost and revenue structures.

A similar nonmonotonic lending response to capital regulation was also observed in De Nicolò, Gamba, and Lucchetta (2014), who formulate a discrete-time partial equilibrium model of a bank exposed to credit and liquidity risks, and subject to financing frictions. Moreover, in a general equilibrium setting, Shaw, Chang, and Chen (2013) also revealed such a nonmonotonic response in a model with similar cost assumptions to ours. The more recent works of Begenau (2019) and Bahaj and Malherbe (2019) also demonstrate various economic mechanisms through which an increased capital requirement may lead to increased bank lending. In Begenau (2019), the equilibrium pricing of bank debt can reduce a bank's funding cost as capital requirements increase, allowing for the expansion of credit. Alternatively, Bahaj and Malherbe (2019) describes a *forced safety effect*, which can lead to increased lending as banks are required to hold more capital. Importantly, the increased lending in Bahaj and Malherbe (2019) is not due to an overall decrease in funding costs (as in Begenau, 2019), but a decrease in the *marginal* funding cost as capital requirements are increased. Taken together, these results suggest that a careful analysis of a bank's cost and revenue flows would be appropriate for a regulator to fully understand a bank's likely lending response.

In modelling a dynamic bank we are also able to explore the effect of loan-loss uncertainty on the interplay between a bank's precautionary (static) controls, namely the holding of an equity

[4]We emphasise that, within our model, the reduction in bank profitability for higher capital requirements is driven both by a tighter constraint on the bank's allowable balance sheet composition and by an increased likelihood of costly external financing.

buffer so as to help absorb future defaults, and responsive (dynamic) controls, which can only act after a default event has occurred, i.e., the issuing of new equity or loans. In this regard, we find that, taking the total expected default size as fixed, the largest precautionary equity buffers are chosen by banks that are exposed to only moderate loan-loss risk. This is because banks exposed to small but frequent (low-risk) loan losses are able to issue new loans at roughly the same rate as their small loan defaults occur, therefore avoiding the need to hold precautionary equity. On the other hand, banks exposed to large but infrequent (high-risk) loan losses discount the impact of these rare events so much that they have little effect on their optimal behaviour and hence the desire to hold a (costly) precautionary equity buffer. This nonmonotonic behaviour of optimal capital ratios to changes in loan default variance highlights the need to consider alternative/further risk metrics when evaluating the stability of a bank—here the calculation of the probability of early closure comes into its own.

Finally, we find that the capital adequacy ratio that minimises the probability of bank closure (and that which maximises loans) is fairly robust to changes in loan default variance. This allows our 'optimal' value to be easily approximated using a deterministic version of our model and a simple formula to calculate its value is provided. Further investigation of this robustness may also contribute to the discussion on the subjective risk-weighting of assets in the calculation of bank capital ratios.

The remainder of the paper is structured as follows. In Section 2 we detail the underlying bank model and its mathematical representation. Section 3 presents the initial analysis and calibration of our model and Section 4 outlines our full results and subsequent discussion. The work is concluded in Section 5.

2. The Banking Model

The problem we consider has two participants: the bank and the regulator, each of whom have different objectives. The bank's objective is monetary, where we assume they seek to maximise their market value of equity V. To maximise this value they must optimally control the bank's operations, where we prescribe four controls: the rate of new loan-issuance l, the rate of dividend payment u, the rate of new (costly) equity-issuance f, and the endogenous closure time ν. In contrast, the regulator's objective can be manifold and even non-monetary, and thus we shall explore the impact of capital requirements upon several key banking metrics, most notably the size of the bank's (risky) loan portfolio L_t, and the probability of bank closure P. We shall give particular attention to the probability of bank closure because the regulator's primary goal is to maintain the stability of the financial system, and to help achieve this they must reduce the likelihood, and impact, of bank insolvency. Their control in this regard is the capital adequacy ratio ω, and it is designed to ensure a minimum equity 'buffer' is in place to help protect the bank's longevity against future loan losses. Within the construct of our model, we define the capital adequacy ratio as the minimum ratio between the book value of shareholders' equity E_t and the book value of the loan portfolio L_t (Heid, 2007).[5] The implication of this is that for all points in operational time the

[5]For further background information about this ratio, and its importance in the various Basel Accords, we refer readers to Hull (2018), Chapters 15 and 16 in particular, as a starting point.

bank's balance sheet must satisfy

$$\frac{E_t}{L_t} \geq \omega, \tag{1}$$

where equality is reached along the regulatory line $E_{\min} = \omega L$. Should this condition be violated, but the bank remain solvent, the bank will be forced by the regulator to raise the appropriate amount of new equity, which we denote by f_ω. For example, in 2013 the UK's Barclays Bank had to issue some £5.8Bn worth of new shares in order to meet the Prudential Regulation Authority's capital requirements (Barclays, 2013). If, however, the bank becomes insolvent (such that $E_t < 0$) the regulator will place the bank into resolution and the shareholder's control will be removed.

As well as making sure the capital requirement is not set too low, the regulator must also ensure they do not increase the requirement too far, for this may increase the likelihood of the bank deciding to close endogenously (due to insufficient return on equity for shareholders) and also impact upon the lending volume to the loan market. For these reasons, we shall pose the regulator's objective to be the minimisation of the overall probability of bank closure, subject to constraints upon bank lending levels.[6]

The above motivations for the bank's and regulator's objectives can be framed as a joint optimal control problem. Formally, this problem can be written as:

$$V^* = V(X \mid l^*, u^*, f^*, \nu^*) = \max_{l,u,f,\nu} \left\{ V \mid \frac{E_t}{L_t} \geq \omega^*, \ \forall t \right\} \tag{2}$$

with

$$P^* = \mathbb{P}(\nu^* < T_r \mid \omega^*) = \min_{\omega} \{ \mathbb{P}(\nu < T_r \mid l^*, u^*, f^*, \nu^*) \mid L^* \in \mathcal{L} \}, \tag{3}$$

where we use asterisks to denote optimal values, $\nu \in [0, T_b]$, where T_b is the operational time horizon of the bank (which could justifiably be infinite), T_r is the time horizon over which the regulator wishes to monitor, $\mathcal{L}$ is the set of admissible loan portfolio size constraints, and $X \in \mathbf{R}^n$ is a stochastic process that will define the state-space of the underlying uncertainty dynamics in our model. To extract the solution to (2) and (3), we must first expound our model inputs further in order to define X, which we do so by considering the bank's balance sheet.

On the asset side of the balance sheet, the bank's total assets are given by the sum of the loan portfolio L_t and the current size of the cash reserves R_t. The revenue from these assets come from the averaged interest paid upon loans (including fees charged), r_L, and the interest on cash assets r_R. The income within a small time dt is therefore $(r_L L_t + r_R R_t)dt$. Furthermore, we assume the loan book is composed of plain vanilla loans (i.e., unsecured cash loans), which are being repaid at a rate a with new loans being issued at a (controlled) rate l. However, because of the default risk there is always the possibility that portions of this portfolio are not repaid. To effectively capture this loan-default dynamic, we assume the loan-default uncertainty is in the form of a fat-tailed distribution, with small but frequent defaults together with large but infrequent defaults; Elsinger, Lehar, and Summer (2006) categorises such loan-losses into two forms: *fundamental defaults*

[6] This risk-minimisation is found without loss of generality, for we shall explore these bank metrics over the whole range or capital requirements, and thus one could use our results to explore the impact of alternative regulatory objectives.

(small, relatively frequent and manageable) and *contagious defaults* (large, infrequent and hard to mitigate). We can therefore model these loan defaults as a jump process that is composed of two parts: Firstly, a known percentage μ that represents the expected size of the frequent loan defaults and secondly, a compound Poisson jump process Q_t, which represents large but infrequent loan defaults. The frequency of these jumps is denoted by λ and the individual jump size distribution is given by h with mean K, making the total expected percentage jump λK. Analogous write-down problems are considered within the actuarial sciences, e.g. Avanzi (2009); Lin and Pavlova (2006); Embrechts, Kluppelberg, and Mikosh (2003). Taken together, this means we can write the total loan portfolio process as

$$\frac{dL_t}{L_t} = (l - a)dt - (\mu + \lambda K)dt - d\hat{Q}_t, \tag{4}$$

where $d\hat{Q}_t = dQ_t - \lambda K dt$ is the compensated compound Poisson process (with zero mean). For the jump size distribution h, we specify an exponential distribution given by

$$h(z) = \frac{A}{1 - e^{-A}} e^{-Az}, \tag{5}$$

where A is a positive constant and $z \in [0, 1]$ in order to bound the loan-losses between 0–100%. The expected jump size is thus $K = [1 - e^{-A}(1 + A)]/A(1 - e^{-A})$.[7]

To define the cash-flow process dR_t, we must consider each possible outgoing and incoming cash amount. The incoming cash is generated by revenues, loan repayments, and any equity raised. The raised equity comes in two parts: the rate of equity endogenously raised f; and the rate of equity forcibly raised due to regulation f_ω. This means that within dt, the total incoming cash is $(r_L L_t + r_R R_t + a L_t + f + f_\omega)dt$. The outgoing cash is generated through new loans $lL_t dt$, bank running costs $\epsilon dt = \epsilon(X)dt$, controlled dividend payments udt and interest paid on the total deposit volume D_t at a rate r_D. The cash flows must also account for the net change in deposit volume dD_t. All of these payments allow us to write the total cash-flow process as

$$dR_t = (r_L L_t + r_R R_t + a L_t + f + f_\omega)dt - (lL_t + \epsilon + u + r_D D_t)dt + dD_t. \tag{6}$$

On the liability side of the balance sheet are the total bank deposits D_t, and it is also where the shareholders' equity E_t features. The shareholders' equity may be thought of as the amount of money that would be returned to investors upon immediate closure of the bank (minus any liquidation costs). This notion is quantified by the well-known balance sheet criterion which holds for any instance in operational time:

$$L_t + R_t = D_t + E_t. \tag{7}$$

Any deviations in L_t caused by the default of loans will be written off against the value of E_t, and if the size of the default is large enough for E_t to become negative, the bank would be insolvent

[7] We do not explicitly specify a value for A, but instead specify the (observable) total expected jump size λK and the frequency of jumps λ (allowing for the inference of A).

and thus not hold enough assets to repay all deposits. Making use of (4), (6) and (7) allows us to write the dynamics of the shareholders' equity as

$$dE_t = (\Phi_t - u + f + f_\omega)dt - L_t d\hat{Q}_t. \tag{8}$$

where Φ_t is the expected profit process:

$$\Phi_t = r_L L_t + r_R R_t - r_D D_t - \epsilon - (\mu + \lambda K)L_t. \tag{9}$$

From looking at the second term on the right hand side of (8), one can clearly see how the shareholders are exposed to uncertainty, as each loan-loss is subtracted (written-down) from its value.

*** Insert Figure 1 about here ***

Figure 1 provides a graphical illustration of how cash flows into and out of the bank in our model. These prescriptions of the balance-sheet components allow us to define the dimensions of the state-space X in which we must solve our model. At first glance it might appear that there are five dimensions to consider: (t, L, E, D, R). Fortunately, the balance sheet equation (7) enables us to substitute out one of these terms and reduce the number of dimensions to four. In addition, since this paper is primarily dealing with the effects of loan-loss uncertainty and capital requirements, rather than liquidity requirements, we choose to make the simplifying assumption that the deposit volume D_t remains constant, meaning $dD_t = 0$. While this assumption might be considered restrictive, we feel that this is a necessary first step. This assumption may also be justified in part by noting that bank liquidity risks are diminished by the presence of central bank liquidity schemes (e.g., Cecchetti and Disyatat, 2010) and deposit insurance schemes (e.g., European Commission, 2014b; Diamond and Dybvig, 1983), which help reduce the impact of deposit variance on a bank's optimal decision making.

With the assumptions above, and by substituting out R from our equations using (7), the dimensional space this paper shall deal with is:

$$X = (t, L, E). \tag{10}$$

Furthermore, the expected profit process (9) can be re-written as

$$\Phi_t = \Phi(L_t, E_t) = r_R E_t + (r_L - r_R)L_t - (r_D - r_R)D - \epsilon - (\mu + \lambda K)L_t. \tag{11}$$

To capture the endogenous closure mechanism, we note the optimal timing is when the owners of the bank are indifferent between continuing ownership or receiving their underlying equity back. The timing for when this optimal decision is taken is solved as part of the model, for it represents a free-boundary (Brennan and Schwartz, 1985). Denoting this optimal decision line by $L_t = C^*(t, E_t)$, we can write the endogenous closure criteria as

$$V = E - \phi L \quad \text{on} \quad L_t = C^*(t, E_t), \tag{12}$$

where ϕ denotes a constant liquidation cost of the bank's risky assets (L).

Finally, we define how capital regulation will affect a solvent bank. When a loan default causes a violation of condition (1), we assume the regulator will observe this and force the bank to raise enough equity through a share issuance (at market value) so that the capital adequacy condition is satisfied. This will thus bring in a cash amount $E_{\min} - E_t = \omega L_t - E_t$ (if $E_t/L_t < \omega$). In other words, at the current balance sheet location (L_t, E_t), new shares are forcibly issued to raise $\max\{\omega L_t - E_t, 0\}$.

However, the raising of external equity necessarily comes with a cost. Similar to Bolton, Chen, and Wang (2011), we model in reduced form the costs associated with asymmetric information (Myers and Majluf, 1984), agency conflicts from managerial incentives (Jensen and Meckling, 1976), as well as direct transaction costs (Calomiris and Himmelberg, 1997): whenever the bank chooses (or is forced) to raise new equity it will incur a cost ψ, which is proportional to the amount of equity raised, i.e. $\psi = \psi_0(f + f_\omega)$, for some constant ψ_0. Due to this cost, we note that in some circumstances (i.e., after a particularly large write-down) the bank may choose to endogenously close the bank rather than refinance if the cost of refinancing is deemed too high when compared to the bank value after re-capitalization (Bolton, Chen, and Wang, 2011; Hugonnier and Morellec, 2017). Note that, while a fixed cost can also be included in our modelling, it would only serve to increase the (already large) parameter space and computation time, without altering our economic and modelling insights.

To understand how the various bank and regulatory controls interact with one another, the schematic diagram in Figure 2 shows how the bank will operate for differing compositions of (L_t, E_t). As can be seen, the model implies the existence of five key regions in which the bank may move as loan defaults randomly occur—note that when a default occurs the bank will move position by reducing both E_t and L_t by the amount that is defaulted, i.e. a south-west (225 degree) jump. In Region 1 the bank is fully operational and is not violating any capital requirements, meaning the bank can freely choose their own dividend, loan and refinancing policies. However, if a default moves the bank from Region 1 into Region 2, then the bank will be in breach of their capital adequacy ratio (i.e., $E_t < E_{\min}$) and the regulator will immediately force them to raise enough equity to meet the required level ωL_t. Yet if a large enough default occurs, the bank would enter Region 3, which is where the shareholders' equity base is reduced to below zero. Here the bank is technically insolvent, unable to repay all bank deposits, and is placed into resolution. The second closure mechanism is represented by the bank crossing the line $L_t = C^*(t, E_t)$ and moving into Region 4, which is where it is optimal for the bank to endogenously close down. This occurs when the bank is making enough of an operational loss that it is better to immediately return all remaining value to shareholders, than it is to continue making losses. Region 5 is the cash-flow insolvency region, where the bank is holding negative cash reserves. We refer to the line $E_t = L_t - D$ as the *zero-cash line* since along this line $R_t = 0$ and the bank cannot cover any increase in outgoing withdrawals. Our assumption of a constant deposit volume however will ensure that the bank will never enter into this region.

*** Insert Figure 2 about here ***

2.1. The Banker's Equation

We assume that all agents are risk-neutral and discount cash flows at a constant rate ρ (cf. Hugonnier and Morellec, 2017). The market value of equity V^* is therefore defined as the expectation of the discounted future dividend payments minus future equity issuance payments, plus the equity value that is returned to the shareholder should the bank close (Bolton, Chen, and Wang, 2011). Using the terms defined in the previous section, this allows us to re-write the bank's objective, for a given initial balance sheet (L, E), as:

$$V^* = \max_{l,u,\nu,f} \left\{ \mathbb{E}_t \left[\int_t^{\nu} e^{-\rho(s-t)} (u - f - f_\omega - \psi) ds + (E_\nu - \phi L_\nu) e^{-\rho(\nu - t)} \right] \right\}, \tag{13}$$

which holds for any given capital adequacy ratio.[8]

By referring to Appendix A one can see the above expectation may be re-cast as the solution to the partial integro-differential equation (PIDE) given by

$$\begin{aligned} &\frac{\partial V}{\partial t} + \lambda \int_0^1 h(z) \left[V(t, L - zL, E - zL) - V(t, L, E) \right] dz - \rho V \\ &\quad + \max_{l,u,f} \left\{ (\Phi - u + f + f_\omega + \lambda K L) \frac{\partial V}{\partial E} + (l - a - \mu) L \frac{\partial V}{\partial L} + u - f - f_\omega - \psi \right\} = 0, \end{aligned} \tag{14}$$

which is subject to the boundary conditions defined by

$$\begin{aligned} &\frac{\partial V}{\partial E} \to 1 \quad \text{as} \quad E \to \infty, \\ &V = E - \phi L \quad \text{on} \quad L = C^*(t, E), \\ &V = 0 \quad \text{on} \quad E \leq 0, \end{aligned}$$

where for notational convenience we have dropped the asterisk for V. In regard to the prescription of these boundary conditions: the first condition tells us that a bank's cash flows become less important to the valuation if it already holds extremely large levels of shareholder equity E; the second boundary condition states the value returned to shareholders upon endogenous closure of the bank; and the third boundary condition means that when the insolvent bank is placed into resolution, all shareholder value is removed. We also impose a final boundary condition that tells us the bank shall continue to operate provided they have avoided the two previous closure mechanisms, i.e. that the bank shall run perpetually and thus $T_b \to \infty$. We may represent this via the condition

$$\frac{\partial V}{\partial t} \to 0 \quad \text{as} \quad T_b \to \infty. \tag{15}$$

Note that this condition implies that $V(t, L, E) = V(L, E)$ and $C^*(t, E) = C^*(E)$, hence a stationary solution is sought.

[8]The notation $\mathbb{E}_t$ represents expectation conditional on the value of the uncertainty process at time t, i.e. $L_t = L$ and $E_t = E$.

2.2. The Regulator's Equation

The probability of overall closure may be thought of as the expected proportion of all future scenarios that lead to the closure of the bank before time T_r. Taking the bank's optimal strategy as given, the closure probability may then be written as

$$P = \mathbb{E}_t \left[I_{\{\nu \wedge \tau_0 < T_r\}} \,|\, l^*, u^*, f^*, \nu^* \right], \tag{16}$$

where $\tau_0 := \inf\{s \geq t \,|\, E_s \leq 0\}$ denotes the insolvency time of the bank. Hence $I_{\{\nu \wedge \tau_0 < T_r\}}$ is one if closure of the bank occurs before T_r, and zero otherwise. As with the banker's equation (14), one can refer to Appendix A to see that (16) corresponds to the solution of the equation

$$\begin{aligned} &\frac{\partial P}{\partial t} + \lambda \int_0^1 h(z) \left[P(t, L - zL, E - zL) - P(t, L, E) \right] dz \\ &\qquad + (\Phi - u^* + f^* + f_\omega + \lambda K L) \frac{\partial P}{\partial E} + (l^* - a - \mu) L \frac{\partial P}{\partial L} = 0 \end{aligned} \tag{17}$$

with boundary conditions defined by

$$\begin{aligned} &P = 0 \quad \text{when} \quad t = T_r, \\ &\frac{\partial P}{\partial E} \to 0 \quad \text{as} \quad E \to \infty, \\ &P = 1 \quad \text{on} \quad L = C^*(E), \\ &P = 1 \quad \text{on} \quad E = 0. \end{aligned}$$

The first condition indicates that the bank is still operational if it has reached the end of the regulator's time horizon; the second condition implies the probability of closing tends to zero as the shareholder equity base grows extremely large; the third and fourth conditions mean the bank has closed either through endogenous closure or technical insolvency, respectively. The reason why the above equation for P does not contain an explicit minimisation is due to the fact that ω is fixed from the initial time (i.e., it cannot be continuously adjusted). As such, in order for the regulator to influence the size of P they must utilise the bank's optimal control strategy (l^*, u^*, f^* and C^*) for any given ω, and then calculate that particular probability of early closure. The value of ω which produces a minimum value of P, whilst also satisfying any lending targets, may be considered the optimal control for the regulator.

3. Model Analysis and Calibration

Within this section we provide some initial analysis of our banking model. We first discuss the importance of the bank's optimal balance sheet location and provide intuition on its likely location by considering the (analytically tractable) solution to our model when the loan-loss variance is removed. We then provide details of our solution methodology in the stochastic setting and discuss the calibration of our model parameters to observable banking metrics.

3.1. Stationary Points

An important concept within the forthcoming analysis is the existence of an optimal balance sheet location—referred to as a *stationary point*—which we denote by

$$\eta^* = (L^*, E^*). \tag{18}$$

Once the bank has arrived at such a point, it will be rational to want to remain there. The only way the bank will move from this stationary point is in the event of a default, after which the bank shall go through the process of trying to regain this position as quickly as possible.

Clearly the size of the capital adequacy ratio might alter the location of the stationary point since it alters the region in (L, E) space where the bank can remain and thus we observe that $\eta^* = \eta^*(\omega)$. As such, it makes sense for us to focus our examination of the bank's valuation, loan issuance, and probability of early closure at η^* for each different ω, for otherwise we would not be comparing one rational bank situation with another.

The stationary point captures the notion that a bank will typically become more profitable the larger their balance sheet is, but only up to a certain critical size of loan portfolio. This critical size will be determined either by competition laws that place restrictions upon market share or, more commonly, by the cost and risk-structure of the bank. Indeed, as stated by Lindquist (2004) in regard the issuing of new loans: "more extensive screening and monitoring are costly . . . and banks probably balance the cost of and gain from these activities against the cost of holding excess capital." This is to say that for larger loan portfolios, the bank will have to significantly increase their running costs, takeover costs, regulatory burden and/or default risks in order to increase their market share of the banking sector, thereby reducing the marginal benefit of doing so. This reasoning is also supported by the fact that open economies do not have only one bank issuing all loans (for further discussion, see Berger, Klapper, and Turk-Ariss, 2009).

Given the above, we follow Elyasiani, Kopecky, and VanHoose (1995), Kopecky and VanHoose (2004a,b, 2006), and Shaw, Chang, and Chen (2013), for example, and assume a convex cost function which, without loss of generality, is composed of two parts: a constant running cost ϵ_0 (e.g., rent and base salaries), and a second component dependent upon the current size of the loan portfolio (e.g., increased monitoring costs):

$$\epsilon = \epsilon_0 + \tfrac{1}{2}\epsilon_1 L^2 \tag{19}$$

where ϵ_1 is a constant.[9] As will be demonstrated below, this cost function will ensure the existence of a stationary point.

3.2. A Deterministic Bank

To gain some intuition for the bank's optimal behaviour and the likely location of η^* we consider the situation in which the bank's loan-loss variance reduces to zero, yielding a deterministic

[9]Alternatively, this cost function could be modelled as any convex function. The quadratic form chosen here allows us to present some of our key results (such as Proposition 1) in an analytically tractable way. However, the intuition behind the results will hold in more generality.

loan-loss process where loans default at a known continuous rate. A useful way to achieve this, while maintaining the same mean value, is to consider the joint limit $\lambda \uparrow \infty$ and $K \downarrow 0$, such that λK remains constant and finite. In making this assumption, one can appreciate that the bank's optimal policy would be to remain at the stationary point (should it exist) in perpetuity. Hence new loans will be issued at the same rate as the existing loans are repaid and all profits will be distributed as dividends. The bank will also not be required to issue new equity to remain at this stationary point. The bank's optimal controls are therefore given by:

$$l^* = a + \mu + \lambda K, \quad u^* = \Phi, \quad \text{and} \quad f^* = 0.$$

The valuation of the deterministic bank is therefore the present value of the perpetual stream of dividends paid out at the stationary point. In other words,

$$V(L^*, E^*) = \int_t^{\infty} e^{-\rho(s-t)} \Phi(L^*, E^*) ds = \frac{\Phi(L^*, E^*)}{\rho}, \tag{20}$$

for the optimal balance sheet (L^*, E^*).

Equation (20) can now be used to determine the location of η^*. Inspection of (11) reveals that $\partial V / \partial E > 0$ for all balance sheets, i.e., more internal equity would always result in an increased market value V. Therefore, it is clear that simply maximising $V(L, E)$ over L and E is not sufficient to determine the optimal balance sheet location. Instead, it can be seen that the maximisation of V in *excess* of the internal equity required to generate this value, i.e., $\max_{L,E} \{V(L, E) - E\}$, will provide our desired location for the deterministic bank. The quantity $V - E$ may be interpreted as the value of running the bank (for a given balance sheet) and its optimisation seen as the decision of how much initial equity (and loans) to provide the bank with, so that the bank would not be required to change its balance sheet location.

In regard to the existence of such a stationary point, we note that for a given level of equity E, the convex cost function $\epsilon(L)$ will ensure that there is an optimal loan portfolio size. As for an optimal equity level, recalling (11) and differentiating $V - E$ with respect to E yields $r_R/\rho - 1$, from which it is evident that the decision on how much equity to retain within the bank is determined entirely by the spread between the rate the bank earns on its cash reserves r_R and the discount rate ρ. Similar to Bolton, Chen, and Wang (2011), we assume the existence of a *cost of carry* on internal equity and hence that $r_R < \rho$.[10] This implies that $r_R/\rho - 1 < 0$ and the bank will want to keep as little equity inside the bank as possible as increasing E (for L fixed) will reduce $V - E$ for all balance sheet locations. It would therefore be optimal for the bank to pay out dividends until either the regulatory bound upon equity size is met (meaning $E = \omega L$) or there are zero cash holdings left in the bank (meaning $E = L - D$). The exact location of the optimal balance sheet is given by the following proposition.

[10]This assumption appears standard in models with corporate cash holdings (e.g., Kim, Mauer, and Sherman, 1998; Riddick and Whited, 1984) where such costs can arise from agency costs or possible tax distortions (since cash retained within the bank may be taxed at a higher rate than that of personal investors).

Proposition 1. *The optimal balance sheet location (stationary point) of the deterministic bank, $\eta^* = (L^*, E^*)$, is given by*

$$\eta^* = \begin{cases} \left(\bar{L},\ \bar{L} - D\right) & \text{if } \omega < \omega_0, \\ \left(D/(1-\omega),\ \omega D/(1-\omega)\right) & \text{if } \omega \in [\omega_0, \omega_c], \\ \left(\bar{L} + (1-\omega)/\gamma,\ \omega[\bar{L} + (1-\omega)/\gamma]\right) & \text{if } \omega > \omega_c, \end{cases} \tag{21}$$

where $\bar{L} := (r_L - \mu - \lambda K - \rho)/\epsilon_1$, $\gamma := \epsilon_1/(\rho - r_R)$, $\omega_0 := 1 - D/\bar{L}$ and

$$\omega_c := 1 + \frac{1}{2}\left(\gamma\bar{L} - \sqrt{\gamma^2\bar{L}^2 + 4\gamma D}\right). \tag{22}$$

Proof. See Appendix B. □

One can see from (21) that the stationary point is unaffected by capital regulation for $\omega < \omega_0$, but that increasing ω above ω_0 will cause the bank to increase loans and rise up along the zero-cash line. This increase in optimal loan size will continue until ω rises above ω_c, at which point the stationary point will leave the zero-cash line and the loan size will decrease (and cash holdings increase). Hence the convex cost structure of the bank implies that for some critical level of capital adequacy ratio (ω_c) the deterministic bank will optimally choose to hold cash (rather than issuing new loans).

In other words, the deterministic bank will initially respond to increased capital requirements by increasing their loan portfolio size in order to offset the cost of the increased equity required. However, increasing the capital adequacy ratio beyond ω_c will result in a reduced loan portfolio size as the marginal value from increasing loans decreases below the marginal value of holding additional cash. In this regard, we observe that the critical point in (22) increases for a higher expected return on loans ($r_L - \mu - \lambda K$) or a lower expected return on cash (r_R); consistent with the bank's trade-off decision between retaining earnings as cash or issuing new loans.

Another important feature of the bank's optimal behaviour is revealed by considering the value of the deterministic bank at the optimal balance sheet location $V(L^*, E^*)$. Whilst the optimal balance sheet location is known to maximise $V - E$, this maximum is not guaranteed to be large enough to ensure the bank will want to operate at this location (as opposed to closing down and liquidating). Recalling that the rational bank will close down when condition (12) is satisfied we obtain the following result.

Proposition 2. *Defining $\overline{\epsilon}_0 = D(\rho - r_D) + (\epsilon_1\bar{L} + \rho\phi)^2/2\epsilon_1$, $\underline{\epsilon}_0 = D(r_R - r_D) + \epsilon_1\bar{L}^2/2 + \rho\phi\bar{L} < \overline{\epsilon}_0$ and $\overline{\omega} = 1 - D/(\bar{L} + \rho\phi/\epsilon_1)$, the following behaviour is optimal for the deterministic bank:*
(i) If $\epsilon_0 \geq \overline{\epsilon}_0$ the bank should close endogenously for all levels of regulation $\omega \in [0, 1]$;
(ii) If $\epsilon_0 \leq \underline{\epsilon}_0$ the bank should remain operational (i.e., should not endogenously close) for all levels of regulation $\omega \in [0, 1]$;
(iii) If $\epsilon_0 \in (\underline{\epsilon}_0, \overline{\epsilon}_0)$ the bank should close endogenously for $\omega > \omega_{max}$ with $\omega_{max} \in (\overline{\omega}, 1)$ given by

$$\omega_{max} = \left[r_L - r_R - \mu - \lambda K + \rho\phi\left(1 - \sqrt{1 + c}\right)\right]/(\rho - r_R), \tag{23}$$

where $c = 2\epsilon_1(\epsilon_0 + (r_D - r_R)D)/\rho^2\phi^2$.

Proof. See Appendix C. □

Proposition 2 reveals the importance of the bank's cost structure on its optimal closure decision. If costs become too high then the bank will never choose to operate and if they are too low, they will always operate. More importantly however, for intermediate costs we observe that increasing the capital adequacy ratio ω above a critical threshold will induce an otherwise profitable bank to close endogenously. A result evident from the fact that internal equity is expensive and increasing the capital adequacy ratio forces banks to increase their internal equity level.[11]

The above analytical results for the deterministic bank provides intuition for the economic trade-offs at play when determining the bank's optimal behaviour. Importantly, we will see this intuition carry over to the stochastic setting where analytical formulae are not available due to the complexity of the banking model. Indeed many of the same qualitative feature of the deterministic model are present when variance is added; particularly in regard to the banks lending behaviour (see Figure 3). Additional intuition will also be required in the stochastic setting however, since, for precautionary reasons, it is expected that the bank should endogenously select a capital ratio that is higher than the regulatory requirement so as to self-insure against the default risk and to avoid costly regulatory interference (Valencia, 2016). However, we shall revisit this discussion when interpreting our results in Section 4.

3.3. Solution Methodology and Numerical Scheme

We now proceed to describe in detail our solution methodology for the full (stochastic) banking model. In the stochastic setting we must solve equation (14) numerically to obtain the bank's optimal valuation and strategy, and subsequently solving equation (17) will determine the closure probability associated with this strategy.

To do this we first observe that the maximisation term of (14) is linear in the controls l, u and f, and thus it is well known that these optimal controls will be of bang-bang (all or nothing) type (see Fleming and Rishel, 1975). This means that the bank shall try to move to a maximal point as quickly as possible by choosing the largest/smallest values of l, u or f. The lower bounds on these controls are assumed to be zero (i.e., $l, u, f \geq 0$) and u and f are assumed to be unbounded from above.[12] However, due to the effects of customer demand, we choose to impose a finite upper bound on l, implying that $l \in \{0, \bar{l}\}$. The above considerations therefore imply that the bank's freely operational region (Region 1 in Figure 2) will be optimally split into at most four sub-regions: loan issuance; dividend payments; equity issuance; and a region of inaction.

The precise criteria for issuing loans at the assumed maximum rate $\bar{l}$ is found by differentiating the maximisation term within (14) by l, and is given by

$$l^* = \bar{l} \quad \text{if} \quad \frac{\partial V}{\partial L} > 0. \tag{24}$$

[11] For our calibrated parameters used in the following section, $\omega_0 = -0.75 < 0$ (hence the interval $[0, \omega_0)$ is empty), $\omega_c = 0.151$ and $\omega_{max} = 0.391$. Note that the latter exists since $\epsilon_0 = 0.45 \in (\underline{\epsilon}_0, \overline{\epsilon}_0) = (-0.83, 1.68)$.

[12] Note that this latter point does not generate any conceptual problems since there is still a bound upon the total *amount* of new dividends and equity issued at any given point in time. In other words, whilst the rate of issuance can be unbounded, the length of time over which it occurs will be infinitesimal, resulting in a finite limit.

This criteria reflects the intuitive idea that if more value can be generated by issuing new loans, then the bank should do so as fast as it can. If the criteria is not met then the bank will issue no new loans, meaning $l = 0$. An identical argument can be made regarding the payment of dividends, where one must compare whether it is better to increase the size of the risky equity base E or issue safe dividends, leading to the criteria

$$u^* = 0 \quad \text{if} \quad \frac{\partial V}{\partial E} > 1, \tag{25}$$

where dividends will be paid out at a maximum rate if this criteria is not met. Likewise for the issuance of new equity, where the bank must decide whether the cost of increasing the size of the equity base outweighs the risk posed by its current size, we arrive at the criteria

$$f^* = 0 \quad \text{if} \quad \frac{\partial V}{\partial E} < 1 + \psi'(f) = 1 + \psi_0. \tag{26}$$

Should this inequality become breached the bank issues just enough new equity to regain equality.

To determine these regions, equations (14) and (17) need to be solved numerically and we do so via a novel implementation of the semi-Lagrangian numerical scheme (Forsyth and Labahn, 2007). This scheme is perfectly suited to solving optimal control problems in which a stored quantity is to be managed within the presence of uncertainty: such as storing gas (Chen and Forsyth, 2007), mining gold (Evatt et al., 2011) or, in our particular case, controlling an equity buffer. The presence of the first-order derivatives in L, E and t within equations (14) and (17) means that care must be used when constructing a numerical scheme. This is because such derivatives can produce errors which propagate through the scheme and thus produce inaccurate results. This difficulty can be overcome by employing a semi-Lagrangian numerical scheme.

In this particular case we choose an implicit-explicit finite-difference scheme so that the optimisation of l, u and f may be calculated from known quantities, vastly reducing the amount of calculations required. The Langragian style derivative may be written as

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + \frac{dL}{dt}\frac{\partial V}{\partial L} + \frac{dE}{dt}\frac{\partial V}{\partial E} \tag{27}$$

and in order to calculate this quantity with finite difference we may write

$$\frac{DV}{Dt} \approx \frac{V(t + \Delta t, L^*(t + \Delta t), E^*(t + \Delta t)) - V(t, L, E)}{\Delta t} \tag{28}$$

where

$$L^*(t + \Delta t) = L_t + (l - a - \mu - \lambda K)L_t \Delta t, \tag{29}$$

and

$$E^*(t + \Delta t) = E_t + (\Phi_t - u + f + f_\omega)\Delta t. \tag{30}$$

Given an equally-spaced fixed grid in L and E we can use linear interpolation to calculate the value of $V(t + dt, L^*, E^*)$, where the values of L^* and E^* are first-order estimates of the position of L and E at time $t + dt$. Since the choice of l, u and f are made at time t this part of the problem is dealt with implicitly. For simplicity, the integral part of the equation is estimated explicitly using the value function at time $t + dt$ and the trapezium rule combined with interpolation where necessary.

To take account of the regulatory control of recapitalisation through equity issuance, we efficiently encode the optimisation of f by noting that both the new and existing shareholders maintain fair value in this recapitalisation process (i.e., where ownership dilution is offset by an increase in the bank's capital reserves). Therefore, when raising an amount of new equity, E_n, we have

$$V(t, L, E) = V(t, L, E + E_n) - E_n - \psi(E_n). \tag{31}$$

This condition is equivalent to the issuance formulation within (26), in which it is implicitly assumed that the value function V is smooth over an equity issuance (Bolton, Chen, and Wang, 2011). To find how issuance is done optimally, one can differentiate through the above equation with respect to E_n to yield

$$\frac{\partial V(t, L, E + E_n)}{\partial E_n} = 1 + \psi'(E_n). \tag{32}$$

The solution to this equation provides the optimal amount of new equity E_n^* and hence the location of the optimal equity issuing boundary (as E_n^* shall vary with L). Furthermore, since costs are proportional to the amount of equity being raised, one can see that it is optimal for the bank to raise new equity the moment internal equity drops below the optimal financing line defined by (32). Using these facts significantly reduces the numerical computations for optimising f, since the optimal bank value for $E < \omega L$ must satisfy (31) with $E_n = E_n^*$.

More details on the numerical scheme, grid checks and convergence analysis are available from the authors upon request.

3.4. *Model Calibration*

Before we present our main results, we outline the calibration of the model parameters so as to match observed bank characteristics. Without loss of generality we take $D = \$100$Bn to set the scale of the bank. We set $r_R = 1.0\%$ to be consistent with low interest rate environment in many developed countries over the last 10 year. We also set $\rho = 3.5\%$ to ensure that $\rho > r_R$ which provides a positive cost of carry on internal equity (Bolton, Chen, and Wang, 2011). Consistent with the level of deposit costs used by Hugonnier and Morellec (2017) we set $r_D = 2.5\%$. In regard to the financing costs we set $\psi_0 = 5.0\%$ which is consistent with the average underwriter spread for bank seasoned equity offerings between 1996 and 2007 as reported in Boyson, Fahlenbrach, and Stulz (2016) (see also Hugonnier and Morellec, 2017). We also set the liquidating costs to be of the same magnitude by setting $\phi_0 = 5.0\%$. For the maximum loan issuance rate we set $\bar{l} = 5.5\%$.

To calibrate the expected loss of the bank's loan portfolio, as represented by $\mu + \lambda K$ in our model, we consider the historical loan loss provisioning of banks, which we believe will give a good indication of a bank's expected write-downs. The FRED economic database[13] provides data on the loan loss reserve (as a ratio of total loans) for all US banks and, while there is some variation (between 1.2% and 3.7%), we calculate the mean from 1984–2019 to be approximately 2%. Hence we set $\mu + \lambda K = 2\%$ in our base-case model. This figure is also consistent with Table 3

[13]See `https://fred.stlouisfed.org/series/USLLRTL`.

in Kovner, Vickery, and Zhou (2014), who report values in the range 1.7–2.9% for the percentage of nonperforming loans.

Next, we try to match two key bank performance ratios, the bank's net interest margin (NIM) and its return on assets (ROA). Matching these two measures will inform our choice of r_L and the cost function ϵ. Since the NIM and ROA ratios depend on the bank's balance sheet location, we choose to calibrate these ratios at the point $L = D$ and $R = 0$ for simplicity. To match the NIM, which is represented by $r_L - \mu - \lambda K - r_D$ in our model, we note that the historical average *NIM* for global banks has varied considerably (see Saunders and Schumacher, 2000). However, the range for most banks has been between 2–4% and so we choose to set this ratio at 3%, which has been the approximate average value for all US banks in recent years.[14] Given our calibrated values for $\mu + \lambda K$ and r_D, the requirement to match the bank's *NIM* yields $r_L = 7.5\%$. When considering the ROA, which is represented by $r_L - \mu - \lambda K - r_D - \epsilon(D)/D$ in our model, we note that De Jonghe and Öztekin (2015) report a mean ROA of 0.8% for their sample of global banks from 1994–2010. This value is also consistent with the average ROA of all US banks obtained from the FRED economic database.[15] To achieve a ROA of 0.8% we must therefore set $\epsilon(D)/D = 2.2\%$.

Given the above considerations, the representative parameter values we employ (unless otherwise stated) are given in Table 1.

*** Insert Table 1 about here ***

4. Results

Within this section we show how the bank will operate under different capital adequacy ratios, calculated from (14), and how the regulator can indeed minimise the probability of overall bank closure, calculated from (17). To reflect the fact that different banks carry different loan portfolio risks, we also analyse how the bank's strategy and closure probability varies under changes in the loan portfolio's loss frequency λ. Finally, we explore the implication of our model for the much discussed influence of capital regulation on bank lending, showing how endogenous lending rates will vary under differing capital adequacy ratios.

4.1. Steady-state Behaviour

Our first set of results are shown in Figure 3, where the stationary point solutions for the bank are plotted as the capital adequacy ratio is varied (dashed lines given by $E_{\min} = 0, 0.1L, 0.2L$, and $0.3L$). Stationary points for both uncertain loan-losses (triangles) and deterministic loan-losses (bold line and circles) are shown, where the total expected default size, $\mu + \lambda K$, remains the same size in both cases. The circles and triangles correspond to increasing ω by increments of 10% (the stationary points increase upwards in E). Intuitively, we see that it is optimal for the deterministic bank to always hold the minimum amount of equity, whereas the inclusion of loan-loss uncertainty

[14] See `https://fred.stlouisfed.org/series/USNIM`.

[15] See `https://fred.stlouisfed.org/series/USROA`.

into the model has the effect of dramatically increasing the amount of equity that is optimally kept on the bank's balance sheet so as to help absorb future write-downs.

Result 1. *The presence of loan-loss uncertainty and costly equity financing creates a precautionary motive for the bank to self-insure with an internal equity buffer.*

Figure 3 also confirms the prediction that there exists a capital adequacy ratio beyond which it is optimal for the bank to move off the zero-cash line and hold cash; ω_c. Here we observe that this point is between 10% and 20% for both the deterministic and stochastic case—recall that $\omega_c = 15.1\%$ from (22). This result indicates that, in the presence of loan-loss uncertainty, the bank will still choose to hold cash in response to high enough regulation.

Result 2. *Diminishing returns to scale on the bank's loan portfolio results in an incentive to hold cash, rather than issue new loans, as regulatory capital requirements increase beyond a certain level.*

*** Insert Figure 3 about here ***

Associated with the stationary points of the risky bank in Figure 3 (filled triangles), are the optimal strategies of the bank to attain these ideal balance sheets in the event of a write-down. These dynamic strategies are shown in Figure 4 for $\omega = 0\%$ (Panel a), $\omega = 10\%$ (Panel b), $\omega = 20\%$ (Panel c) and $\omega = 30\%$ (Panel d). This figure demonstrates clearly the influence of capital regulation on the bank's strategies to dynamically optimise their market value.

*** Insert Figure 4 about here ***

The near-vertical line (shown for lower L in Panel d of Figure 4) is the endogenous closure boundary C^*, to the left of which it is optimal for the bank to close and return all remaining value to shareholders. Furthermore, the region below the regulatory line ($E_{\min} = \omega L$) is the region where the regulator will force the bank to recapitalise. Next, recall that the region between the endogenous closure line, the zero-cash line ($E = L - D$), and above the regulatory line ($E = \omega L$), is where the bank freely operates without regulatory interference. This freely operational region is itself optimally divided into four sub-regions of control: a region where the bank pays dividends as quickly as possible but does not issue equity or any new loans (ⒾA); a region where the bank does not pay dividends or issue new equity, but issues loans at the maximum rate $\bar{l}$ (ⒾB); a region where the bank does not issue loans or pay dividends, but issues new equity (ⒾC); and a region where no action is taken at all, i.e. no dividends are paid and no new loans or equity are issued (ⒾD). We note that not all four regions are observed for a given level of regulation, which is a reflection of how optimal controls respond to varying levels of regulation.

From Figure 4 we observe how the size and shape of all of these regions change as the regulator increases the capital adequacy ratio from $\omega = 0\%$ to 30%; such as the bank transitioning from the zero-cash line to holding a positive amount of cash somewhere between $\omega = 10\%$ (Panel b) and $\omega = 20\%$ (Panel c). Most notably, however, we see the endogenous closure line increases in L as ω increases, thereby shrinking the freely operational region and thus potentially increasing the likelihood of the bank crossing this closure line.

*** Insert Figure 5 about here ***

In Figure 5 we summarise the optimal response of the bank to changing regulation and plot, as a function of ω: Panel (a) the optimal equity level E^* and the corresponding value of the bank $V(L^*, E^*)$; Panel (b) the optimal loan portfolio size L^*; Panel (c) the optimal capital ratio of the bank E^*/L^*; and Panel (d) the probability of bank closure for four different regulatory time horizons ($T_r = 1, 2, 4$, and 8 years).

The first notable feature of Figure 5 is that all four metrics are insensitive to changes in the capital adequacy ratio for small values of ω (up to around 10%). This result can be explained by the bank choosing to optimally self-insure to these levels endogenously, and so we may consider the regulatory constraint to be *non binding* at these levels. In fact, inspection of Figure 4(a) reveals that in the presence of no capital regulation (i.e., $\omega = 0\%$) the endogenous equity issuance region (①c) contains the majority of the forced-equity-issuance region at the 10% level, as observed in Figure 4(b). This indicates that if the unregulated bank were to enter such a region it would have been optimal for them to re-capitalise anyway.

Result 3. *The flexibility of a bank to endogenously raise capital (albeit at a cost) results in a non-binding regulatory capital constraint for sufficiently small values of* ω*.*

Next, we summarise the findings as regulation is increased further, beyond the non-binding region, where we observe clear effects on all four metrics. Firstly, and unsurprisingly, we observe from Figure 5(a) that once the regulatory constraint becomes *binding* the bank optimally holds more equity as the capital requirement increases. The total value of bank also increases, but the difference, $V - E$ (which can be interpreted as the value of running the bank), decreases.

Secondly, recalling the optimal loan portfolio size of our deterministic bank in Section 3.1, we observe from Figure 5(b) that even when variance is included, the same nonmonotonic response of the bank to increasing capital regulation is also evident, i.e., the optimal loan size initially increases and then decreases beyond a certain level—a point that is relevant to regulators seeking to influence market lending volumes through capital controls (e.g., Swiss National Bank, 2012). Further, the result indicates that such nonmonotonic behaviour is not driven by the stochastic structure of our model, but rather the assumptions regarding the bank's profit function Φ. We shall return to this issue in Section 4.3.

Thirdly, Figure 5(c) shows clearly how the bank chooses to optimally self-insure. When the capital requirements are non-binding, the capital ratio endogenously chosen by the bank is around 13.7%, irrespective of the imposed capital adequacy ratio.[16] As regulation is increased further,

[16] We note that even with a capital adequacy ratio of 0%, the bank is still subject to swift resolution arrangements in the sense that the regulator will take over should the bank become insolvent. As such, our risky bank has much incentive to avoid insolvency, explaining the relatively large endogenous equity buffer.

and the constraint becomes binding, the bank responds by increasing their capital ratio, but the amount of this additional equity buffer over that required by regulation actually gets smaller as the regulation becomes tougher. Moreover, we observe that, even for extremely high capital adequacy requirements, the bank still chooses to keep a small amount of additional equity, over and above that required, in order to avoid the dead-weight loss from the costly external financing enforced by regulators.

Result 4. *A higher capital adequacy ratio results in a higher optimal capital ratio, but a lower equity 'buffer' over that required by regulation.*

Finally, we consider the probability of bank closure, as reported in Figure 5(d). Given that the bank optimally chooses to keep less equity than the amount of loans on the balance sheet, it is to be expected that the probability of closure will increase as the regulator's time horizon increases.[17] It is for this reason that we report time-averaged probabilities (P/T_r) for comparison across time horizons. We observe from Figure 5(d) the following result:

Result 5. *A capital adequacy ratio exists for which the probability of early bank closure is minimised.*

Furthermore, the precise location for the minimum probabilities of differing horizons all lie within a surprisingly narrow band of $\omega^* \in (25\%, 30\%)$, with lower values of ω^* for longer time horizons. We also observe that across all time horizons the majority of the possible reduction in closure probability can be achieved by a capital adequacy ratio of around $\omega \approx 15\%$.

4.2. Varying the Loan Portfolio Composition

One way in which a regulator may examine differences in closure risks between banks is to gauge the sensitivity of the closure probability to the loan-loss parametrisation. After all, different banks shall carry different loan-loss risk profiles and thus it makes sense to see how rational bank strategies vary for distinct lending preferences.[18] To assess this dependence, we vary the loan default frequency λ whilst keeping total expected write-downs ($\mu + \lambda K$) fixed. Given this, a decrease in λ corresponds to an increase in K and subsequently an increase in the variance of the loan portfolio (since the variance of our compound poisson process is proportional to λK^2).

*** Insert Figure 6 about here ***

Figure 6 plots the probability of closure where the loan default frequency λ is changed from the base-case value of 0.8 yr^{-1} to 0.25 yr^{-1} (Panel a), 0.5 yr^{-1} (Panel b), 1 yr^{-1} (Panel c), and 2 yr^{-1} (Panel d). Firstly, we observe that the probability of bank closure decreases as the frequency

[17] In fact, taking the limit of $T_r \to \infty$ the bank would be certain to close at some point in time. Mathematically speaking, for infinite time horizons, anything that *can* happen, *will* happen.

[18] Taken further, a bank regulator might even use such a measure to impose changes in the risk-profile of a bank's loan portfolio, so that the closure risks can be reduced to within a certain tolerance.

of defaults λ increases. This reduction in overall closure risk is evident from the observation that an increase in λ, whilst keeping λK fixed, is associated with a reduction in the variance of loan-losses. As the variance is reduced, loan-losses converge towards a deterministic process, allowing the bank to operate with perfect foresight. This then allows the bank to better avoid closure, potentially allowing the regulator to impose less stringent capital controls. This simple experiment also highlights that the expected write-down size ($\mu + \lambda K$) is not a sufficient risk statistic, as the risk of bank closure is also highly dependent upon the variance of the loan losses. The regulator must know a bank's expected size of defaults and the expected loan-loss frequency to be able to determine the risk of bank closure.

Figure 6 also reveals a highly non-linear relationship between loan-loss variance and the minimum probability of closure, where this probability is over ten times smaller for the bank with $\lambda = 0.5$ yr^{-1} than it is for the bank with $\lambda = 0.25$ yr^{-1}, for example. However, despite this we observe the following result:

Result 6. *The value of ω that minimises the probability of bank closure is reasonably robust across differing loan-loss risk profiles (λ) and at different durations (T_r).*

Specifically, the minimum probability for our base-case bank parameters occurs around $\omega \in (25\%, 30\%)$ in all cases. For low values of λ which equates to a high risk scenario, we have higher values of ω around 30% (for $T_r = 8$). In the intermediate risk scenario this level reduces slightly to around 25% for all T_r, before going back up again for less risky scenarios.

Whilst variation of the bank's loan risk profile does not appear to significantly affect the regulator's optimal capital adequacy ratio, its effect on the optimal strategy of the bank, particularly their optimal self-insurance strategy, is somewhat more complex. In this regard, we next look at the bank's optimal balance sheet location and valuation as the loan default frequency λ is varied. Figure 7 plots the bank's optimal capital ratio (Panel a) and value of the bank (Panel b) as a function of λ; for levels of regulation $\omega = 0\%$, 10%, and 20%.

*** Insert Figure 7 about here ***

Figure 7(a) reveals that the bank's optimal capital ratio (E^*/L^*) converges towards the required capital adequacy ratio (ω) as the loan-loss frequency increases. This is consistent with the results for our deterministic bank, outlined in Section 3.1, since as $\lambda \to \infty$ the loan risk profile will become deterministic and defaults will happen near-continuously (but of negligible size). The bank can therefore treat these loan-losses as a fixed (deterministic) running cost, thus allowing them to hold a negligible equity buffer above that required (see Figure 3, circles).

Perhaps the most striking observation from Figure 7, however, is the nonmonotonic response of the bank's optimal capital ratio and valuation to changes in default risk. As λ is decreased (and the variance of loan-losses increase) the bank will start to hold more equity to self-insure against the increasing default risk. However, as λ is decreased further still, the bank's optimal response changes. We observe that a bank exposed to very infrequent but large defaults (very low λ) will in fact choose a lower capital ratio than a bank with more frequent but smaller defaults (a higher

λ). In other words, the bank's risk management of such large but infrequent losses is virtually non-existent. Losses are expected to be so large that when they do occur they are likely to result in insolvency and keeping enough equity in reserve to insure against such losses is simply too expensive for shareholders. The bank rationally prioritises shareholder value over the stability of the broader financial system, because higher revenues (from lower equity levels) in the shorter term are enough to offset the longer term closure risks. This behaviour is a consequence of the limited liability of bank equity (Merton, 1977), which provides the bank with the ability to extract higher rents from riskier loan portfolios. This increase in value is evident from Figure 7(b) as λ decreases toward zero. All of the above implies the following result:

Result 7. *A bank with either very frequent or very infrequent loan-losses will carry a lower capital ratio, whereas a bank with a moderate loan-loss frequency will carry a higher capital ratio.*

This nonmonotonic behaviour of the bank's optimal self-insurance indicates that the risk profile of the bank's loan portfolio has a drastic (and non-trivial) effect on the quantification of the *Taxpayer's Put* (Eberlein and Madan, 2012) and the valuation of deposit insurance schemes (Diamond and Dybvig, 1983).

4.3. Lending Impact

Finally, we consider the effect of capital regulation on optimal bank lending behaviour and assess its robustness across differing loan-risk profiles.

*** Insert Figure 8 about here ***

An explicit plot of the sensitivity of the optimal loan portfolio size as ω is varied is shown in Figure 8 for $\lambda = 0.25, 0.5, 1$, and 2 yr^{-1} (along with the deterministic case $\lambda \to \infty$). As before, we observe an initial increase in the optimal loan portfolio size as regulation increases. However, the bank's convex cost structure ensures that they eventually decide to retain earnings as cash rather than issue them as loans, resulting in reduced lending by the bank beyond some critical level of capital adequacy ratio. This nonmonotonic behaviour is observed for all four loan-risk profiles and, similar to the variation of the early closure probability with λ, we observe that the level of regulation at which the maximum loan size is achieved (~15-20%) remains surprisingly robust to changes in the loan risk-profile. This result indicates that the response of a bank's loan-issuance to capital regulation is driven mainly by our modelling assumptions for the bank's profit function Φ rather than the variance structure of the underlying uncertainty. Such profit functions would therefore require careful analysis from a regulator.

Result 8. *The level of regulation that maximises loan issuance is fairly robust across differing loan portfolio variance levels (λ).*

This apparent robustness of optimal capital regulation to a bank's loan-loss risk profile would be of practical interest to regulators. It is also evidence, perhaps, that a pure leverage ratio could be effective without resorting to the inclusion of more subjective risk-weightings; which can also have unintended consequences (e.g., Wu and Zhao, 2016; Mariathasan and Merrouche, 2014; Le Leslé and Avramov, 2012; Blum, 2008). Further investigation of this question, however, is left for the subject of future research. However, we do recall that the level of capital regulation that maximises loan issuance for the deterministic bank can be calculated analytically from Eq. (22); being 15.1% for our base-case bank. The difference between these two optimal values (15.1% and 15-20%) is therefore attributed to the existence in uncertainty in the model. Notably, the difference between the values of the probability of closure at these two levels is very small, giving some space for the regulator to take account of other considerations without drastically altering the closure risks. One such consideration is that of model uncertainty. Figure 6 reveals an asymmetry in the minimum closure probability, where a small increase from the optimal capital requirement can lead to a large increase in the probability of endogenous closure, whereas a small decrease has a much less pronounced effect upon the probability of insolvency. In the complex and changing world in which banks operate, it could be argued for regulators to err on the side of caution and choose a capital adequacy ratio closer to that provided by the analytical formula, safe in the knowledge that the overall modelled closure risks should alter less from the actual risks than a higher capital requirement would.

5. Conclusions

Within this paper we have derived a continuous-time optimal stochastic control model of a regulated bank that is exposed to loan-loss risk. The bank's objective was to maximise their market value of equity by using four distinct bank controls: new loan and equity issuance, dividend payments and endogenous closure. We suggested that the objective of the regulator was to minimise the overall probability of bank closure whilst also considering the corresponding bank lending volumes. Investigation of the bank's sensitivity to capital adequacy ratios revealed that this probability minimisation could be achieved and hence the bank and regulator could each reach their optimal objectives. Higher capital requirements clearly reduce the probability of bank insolvency, however, the regulator must also be reticent of a bank's operating and financing costs, since, if the bank's capital requirements are set too high bankers may choose to cash out their equity and endogenously close the bank (or part thereof). In simple terms, the regulator must allow bankers to earn a normal profit.

In addition to the probability of bank closure, we were able to assess the likely effects of increased capital requirements (beyond the levels for which we have existing empirical data) on optimal bank lending behaviour. We showed that the optimal bank lending policy may be a non-monotonic function of the capital adequacy ratio. In other words, as the capital adequacy ratio is increased, a rational bank may actually begin to increase the size of their loan portfolio in order to offset the additional cost of regulatory capital. However, the convex cost structure of a bank's loan portfolio will ensure that, beyond a critical capital adequacy ratio, banks prefer to retain earnings as cash in favour of issuing new loans and hence they will thereby reduce the loan portfolio size for sufficiently high levels of regulation. This unintended consequence may act counter to other regulatory objectives, such as encouraging/discouraging lending to the real economy.

As an explicit example of the insights that can be gained from our model we examined the effect of different loan portfolio compositions on bank stability (i.e., probability of bank closure) and lending. We observed (and quantified) the increased stability of a bank if it has small but frequent defaults when compared to a bank with infrequent but large defaults. This point highlighted that only knowing the total expected write-down size for a bank is not a sufficient risk statistic, as one also needs to know the associated variance in the timing of defaults. Furthermore, the bank's optimal self-insurance level was found to be nonlinear in the loan-loss variance, however the optimal capital adequacy ratio that minimised the probability of bank closure and maximised loan issuance appeared to be robust to changes in such variance.

In regard to the direction of future research, the fundamental bank model developed in this paper can be extended to include further banking complexities. For example, one could include deposit volume uncertainty, thereby introducing liquidity risk into the modelling framework, which, while increasing the complexity of the model, would allow for the investigation of the likely impact of new bank liquidity requirements (European Commission, 2014a). Perhaps more interestingly, it would also allow for the evaluation of the interaction of both bank capital and bank liquidity regulations, and the assessment of any unforseen consequences that may arise from this interaction; see Hugonnier and Morellec (2017) for a step in this direction. Another possible extension is to allow the bank to choose between multiple asset classes (such as differing loan portfolios), which may be influenced by the relative risk-weightings on those asset classes (Ediz, Michael, and Perraudin, 1998). Finally, an important extension is to expand our model for a single rational bank to a larger banking network. In modelling a banking system we would be able to include additional bank controls (such as the setting of bank lending rates) and evaluate the likely equilibrium response of increased capital regulation. Our preference for this bottom-up modelling approach is due to the behaviour of a single bank being more objective and parameterisable than that of an entire banking system. By first considering a single bank to the level of complexity modelled in this paper, we are able to provide a reference point for future banking network models.

Appendix A. Jump Expectations

In order to recast the problem described by (13) and (16) in the form of differential equations—more precisely, partial integro-differential equations (PIDEs)—we must employ a particular form of the Hamilton-Jacobi-Bellman (HJB) equation. This equation allows one to convert the following stochastic representation

$$U = \max_{C} \left\{ \mathbb{E}_t \left[\int_t^{\nu} e^{-\int_t^s c_z dz} F ds + G(X_\nu) e^{-\int_t^{\nu} c_z dz} \right] \right\}, \tag{A.1}$$

into a PIDE, where F is a source term, G is the value of U upon first exit of the underlying process X (started at time t) from the solution domain at $\partial\kappa$ at time ν, c is a discount rate, and C is the set of controls. The procedure for turning this into a PIDE relies upon the use of Itô's formula for a jump process (details of which can be found in: Øksendal and Sulem, 2005; Shreve, 2004; Cont and Tankov, 2004). Within this paper, we have the stochastic process defined by $X = (t, L, E)$, meaning that, after referring to (4) and (8), the quantity defined in (A.1) can be seen to solve the following PIDE

$$\frac{\partial U}{\partial t} + \max_{C}\left\{-cU + (\Phi - u + f + \lambda KL)\frac{\partial U}{\partial E} + (l - a - \mu)L\frac{\partial U}{\partial L}\right.$$
$$\left. + \lambda\int_0^1 h(z)\,[U(t, L - zL, E - zL) - U(t, L, E)]\,dz + F\right\} = 0, \tag{A.2}$$

which is subject to the boundary conditions

$$U = G \quad \text{on} \quad X = \partial\kappa.$$

Recall that λ is the jump intensity of loan defaults and h the probability density function of the (un-compensated) compound Poisson process Q. By imposing suitable definitions for F and G, this relatively abstract expectation allows us to construct the PIDEs related to the valuation and probability of early closure of our model bank.

Appendix B. Proof of Proposition 1

Proof. It has already been established that the deterministic bank will optimally keep as little internal equity as possible and therefore the stationary point must exist on either the regulatory line $E = \omega L$ or the zero-cash line $E = L - D$. These two lines intersect at a point (see Figure 3) with loan size $L_I := \frac{D}{1-\omega}$, meaning that the stationary point η^* will either lie along the regulatory line for $L \leq L_I$, or along the zero-cash line for $L \geq L_I$. Hence to determine its exact location, we must perform the maximisation of $V - E$ over loan levels L subject to the constraint that $E = \omega L$ for $L \leq L_I$ and $E = L - D$ for $L \geq L_I$.[19]

To solve this constrained optimisation, we first note that the maximum over the the regulatory line interval (denoted L_r^*) and the maximum over the zero-cash line interval (L_{zc}^*) are given by

$$L_r^* = \min\left\{\bar{L} + (1-\omega)/\gamma, L_I\right\} \quad \text{and} \quad L_{zc}^* = \max\left\{\bar{L}, L_I\right\}, \tag{B.1}$$

where we have defined $\bar{L} := (r_L - \mu - \lambda K - \rho)/\epsilon_1$ and $\gamma := \epsilon_1/(\rho - r_R)$. Recalling that $\rho > r_R$ (and hence $\gamma > 0$) it is evident from (B.1) that there are three (exhaustive) cases: (i) $L_r^* = L_I < L_{zc}^*$; (ii) $L_r^* = L_I = L_{zc}^*$; and (iii) $L_r^* < L_I = L_{zc}^*$. Furthermore, since L_I lies both on the regulatory line *and* on the zero-cash line, the maximum value must therefore occur at $L^* = L_{zc}^*$ for Case (i) and $L^* = L_r^*$ for Case (iii). We can trivially conclude that $L^* = L_I$ in Case (ii).

Next, it is clear that for a given (and fixed) set of banking parameters, the three cases above will map onto three distinct intervals of the capital adequacy ratio ω. We find that Case (i) corresponds to $\omega < \omega_0$, Case (ii) to $\omega \in [\omega_0, \omega_c]$, and Case (iii) to $\omega > \omega_c$; where ω_0 and ω_c are determined by the conditions $L_I = \bar{L}$ and $L_I = \bar{L} + (1-\omega)/\gamma$, respectively. Specifically:

$$\omega_0 = 1 - D/\bar{L} \quad \text{and} \quad \omega_c = 1 + \frac{1}{2}\left(\gamma\bar{L} - \sqrt{\gamma^2\bar{L}^2 + 4\gamma D}\right). \tag{B.2}$$

[19] If $\omega > 1$ then the regulatory line is always greater than the zero-cash line (i.e, $L_I < 0$), hence the stationary point is determined by maximising $V - E$ over L subject to the constraint $E = \omega L$ for all L.

The optimal loan portfolio size for the deterministic bank is therefore given by

$$L^* = \begin{cases} L^*_{zc} = \bar{L} & \text{for } \omega < \omega_0, \\ L_I = D/(1-\omega) & \text{for } \omega \in [\omega_0, \omega_c], \\ L^*_r = \bar{L} + (1-\omega)/\gamma & \text{for } \omega > \omega_c, \end{cases} \tag{B.3}$$

and hence substitution of $E^* = \max\{\omega L^*, L^* - D\}$ provides the optimal equity level and the location of the stationary point η^*, completing the proof. □

Appendix C. Proof of Proposition 2

Proof. From the closure condition (12) we recall that the bank will close down when $V(L^*, E^*) \leq E^* - \phi L^*$, in other words the bank should close if the value of the bank (V) is below the liquidation value ($E - \phi L$). Noting the dependence of the optimal balance sheet location on ω and using (20) we define the function

$$f(\omega) := \Phi(L^*(\omega), E^*(\omega)) - \rho\,(E^*(\omega) - \phi L^*(\omega))\,, \tag{C.1}$$

from which it is evident that the rational bank will close whenever $f(\omega) \leq 0$. Substituting the optimal balance sheet location (21) into f will yield the desired result.

More specifically, defining $\overline{\omega} := 1 - D/(\bar{L} + \rho\phi/\epsilon_1)$, it is easily verified that $f(\omega)$ is: (a) continuous; (b) constant for $\omega < \omega_0$; (c) increasing for $\omega \in [\omega_0, \overline{\omega})$; (d) decreasing for $\omega \in (\overline{\omega}, \omega_c]$; (e) decreasing for $\omega > \omega_c$; hence (f) $f(\omega)$ attains its maximum at $\omega = \overline{\omega}$.

For (i) we evaluate $f(\overline{\omega}) = D(\rho - r_D) + (\epsilon_1\bar{L} + \rho\phi)^2/2\epsilon_1 - \epsilon_0 =: \overline{\epsilon}_0 - \epsilon_0$ and conclude from (f) above that $f(\omega) \leq 0$ for all $\omega \in [0, 1]$ if $\epsilon_0 \geq \overline{\epsilon}_0$.

For (ii) we evaluate $f(1) = D(r_R - r_D) + \epsilon_1\bar{L}^2/2 + \rho\phi\bar{L} - \epsilon_0 =: \underline{\epsilon}_0 - \epsilon_0$ and $f(0) = D(\rho - r_D) + \epsilon_1\bar{L}^2/2 + \rho\phi\bar{L} - \epsilon_0 = D(\rho - r_R) + \underline{\epsilon}_0 - \epsilon_0 > f(1)$ (upon recalling that $\rho > r_R$). We therefore conclude that the minimum of $f(\omega)$ is attained at $\omega = 1$ and hence $f(\omega) \geq 0$ for all $\omega \in [0, 1]$ if $f(1) > 0$, hence $\epsilon_0 \leq \underline{\epsilon}_0$.

Finally, for (iii) we recall from above that if $\epsilon_0 \in (\underline{\epsilon}_0, \overline{\epsilon}_0)$ then both $f(\overline{\omega}) > 0$ and $f(1) < 0$. Since we also know from (d) and (e) that $f(\omega)$ is decreasing over $(\overline{\omega}, 1]$ it is clear that a unique $\omega = \omega_{max} \in (\overline{\omega}, 1)$ exists, above which the deterministic bank will close endogenously. The expression for ω_{max} is therefore obtained by solving the (quadratic) equation $f(\omega_{max}) = 0$. □

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Tables and Figures

Table 1: Representative parameter values of our model bank.

Parameter	Description	Value
r_R	Cash reserve interest rate	1.0% yr^{-1}
r_D	Bank deposit interest rate	2.5% yr^{-1}
ρ	Discount rate	3.5% yr^{-1}
r_L	Average return on risky loans	7.5% yr^{-1}
D	Deposit volume	\$ 100 Bn
a	Loan repayment rate	2.5% yr^{-1}
ϵ_0	Bank running cost	\$ 0.45 Bn yr^{-1}
ϵ_1	Bank loan cost	0.00035 yr^{-1} \$$\text{Bn}^{-1}$
μ	Continuous default rate	0.0% yr^{-1}
λ	Jump frequency	0.8 yr^{-1}
λK	Total expected jump size	2.0% yr^{-1}
$\bar{l}$	Maximum loan issuance rate	5.5% yr^{-1}
ϕ_0	Proportional liquidation cost	5.0%
ψ_0	Proportional equity issuance cost	5.0%

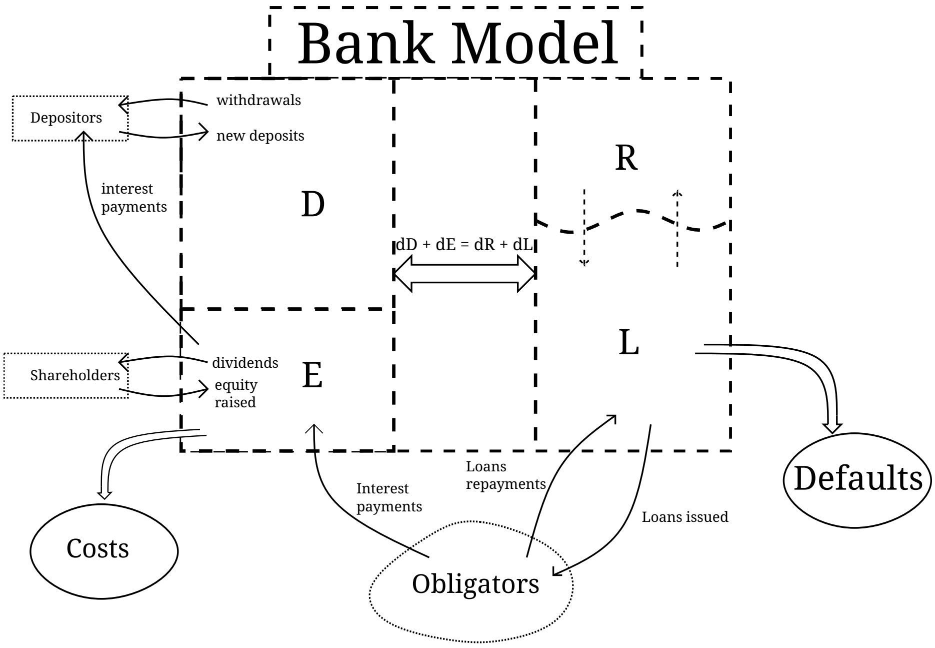


Figure 1: A diagram of the balance sheet and associated cash flows with our bank model.

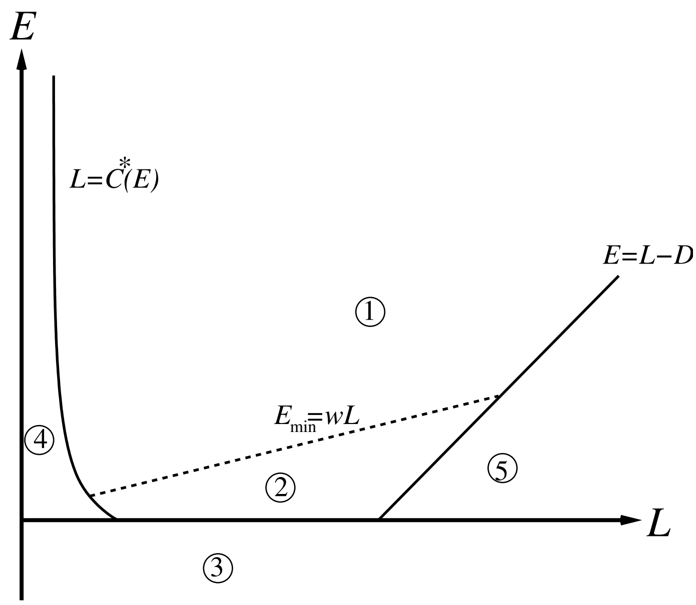


Figure 2: A schematic diagram of the differing regions within (L_t, E_t) (loan, shareholder equity) space where the composition of the bank's balance sheet may move to as loan losses randomly occur.

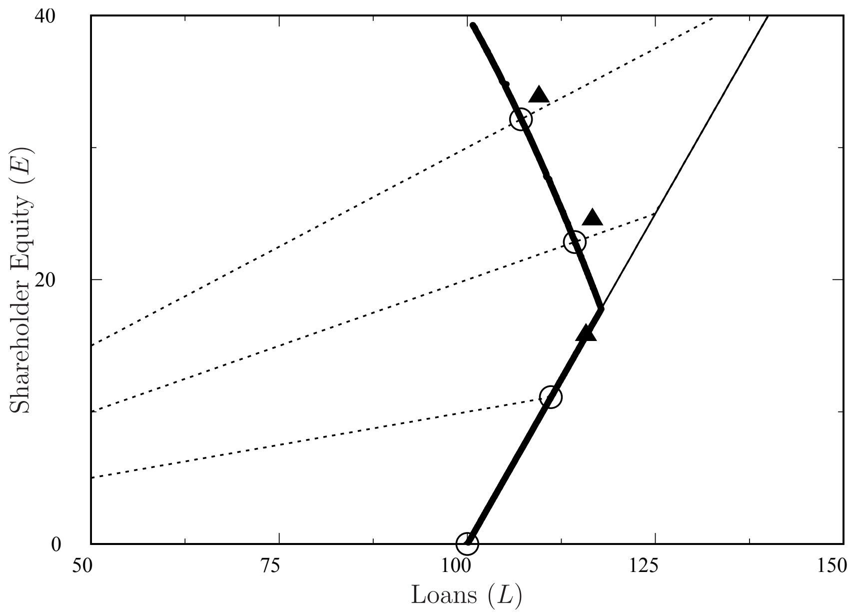
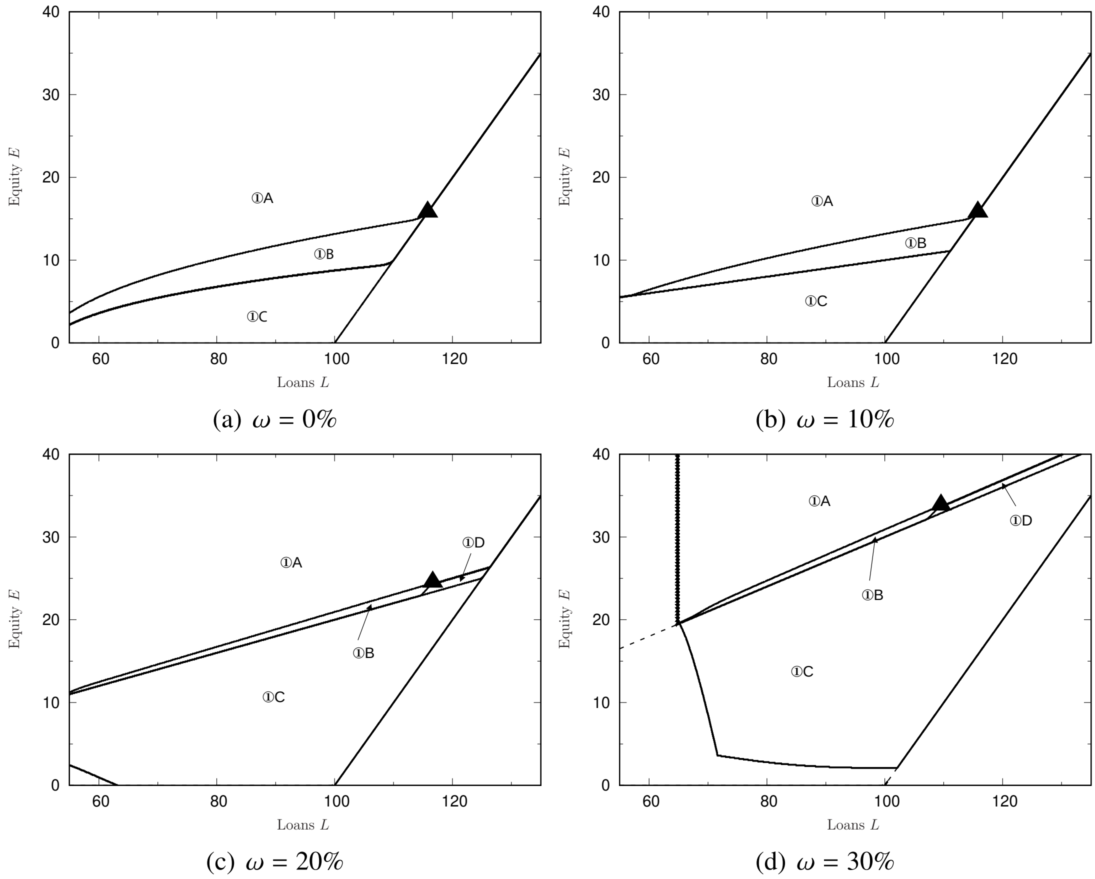


Figure 3: The optimal stationary point with deterministic defaults found analytically (bold line and circles) and with random defaults calculated numerically (triangles). The points increase upwards as the capital adequacy ratio ω is increased: ω = 0%, 10%, 20%, and 30%. We also plot the corresponding regulatory line ($E = \omega L$) with a dotted line. Note that the optimal location for random defaults at ω = 0% and 10% coincide (given by the lower triangle).



(a) ω = 0%

(b) ω = 10%

(c) ω = 20%

(d) ω = 30%

Figure 4: The optimal strategy of the bank subject to a minimum capital adequacy ratio ω. The capital requirements are 0%, 10%, 20% and 30% in subfigures (a), (b), (c) and (d), respectively. Note the freely operational region ① is divided into four sub-regions of control: ①A = pay dividends; ①B = issue loans; ①C = issue new equity; and ①D = no action.

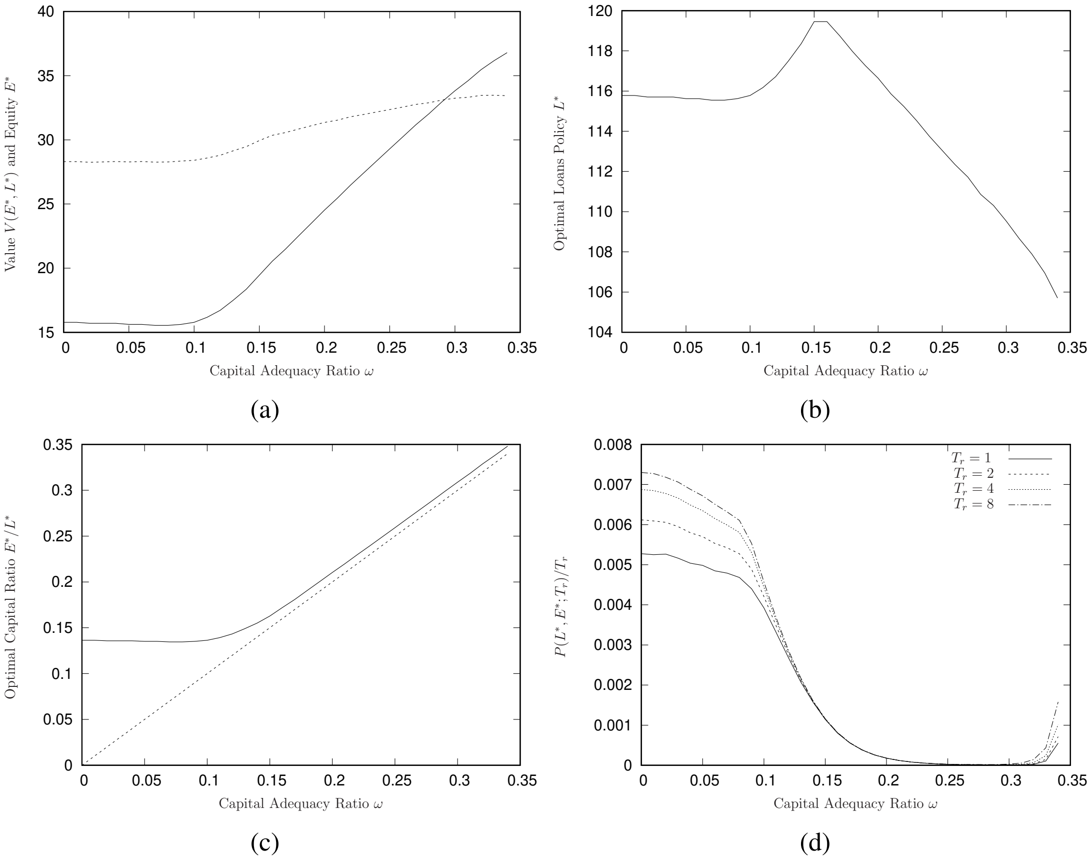


Figure 5: A plot to show how various bank metrics respond to changes in the minimum capital adequacy ratio ω: (a) the optimal equity level E^* (solid line) and the corresponding value of the bank $V(L^*, E^*)$ (dotted line); (b) the optimal loan portfolio size L^*; (c) the optimal capital ratio of the bank E^*/L^* (dashed line corresponds $E/L = \omega$); and (d) the (time-averaged) probability of bank closure for three different regulatory time horizons ($T_r = 1, 2, 4, 8$ years).

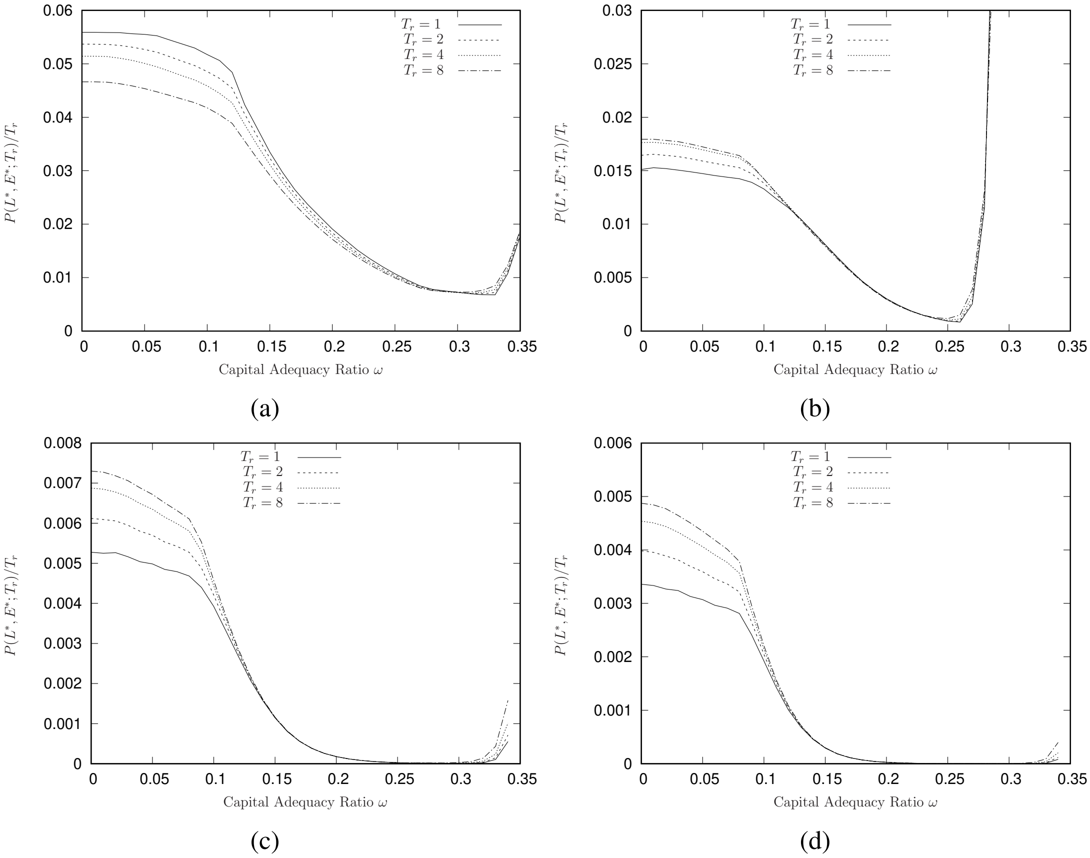


Figure 6: The probability of bank closure (for three different regulatory time horizons) as ω changes. This relation is plotted for four differing values of the loan-loss frequency λ: (a) 0.25 yr^{-1}, (b) 0.5 yr^{-1}, (c) 1 yr^{-1}, and (d) 2 yr^{-1}. Note the change of scale between the subfigures.

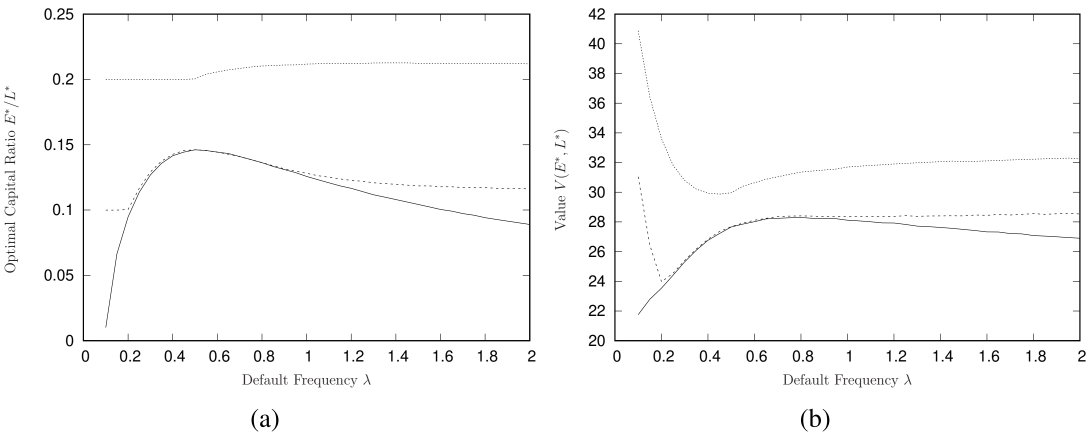


Figure 7: A plot to show how the risk profile (λ) influences the response of the bank's optimal self-insurance strategy for various levels of the capital adequacy ratio ω. (a) The capital ratio, E^*/L^*, and (b) the value of the bank, $V(L^*, E^*)$: solid line (ω = 0%); dashed line (ω = 10%); and dotted line (ω = 20%).

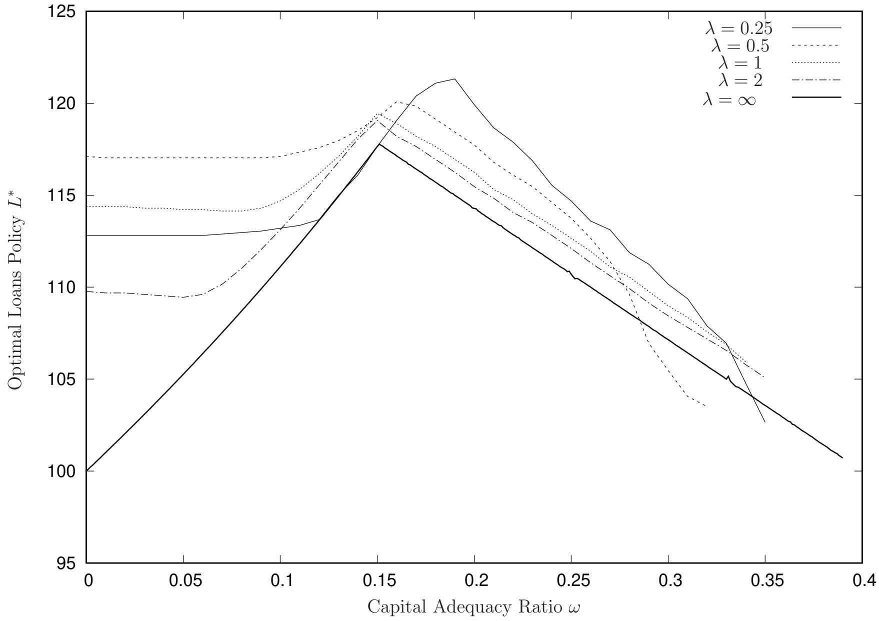


Figure 8: The optimal loan portfolio size as a function of the capital adequacy requirement (ω) for various loan risk profiles λ. The optimal loan size for the deterministic bank is also plotted for comparison.